On Initial-Boundary Value Problem of Stochastic Heat Equation in a Lipschitz Cylinder

TongKeun Chang [1], Kijung Lee [2], Minsuk Yang [1]

[1] Department of Mathematics, Yonsei University
Seoul, 136-701, Korea
[2] Department of Mathematics, Ajou University
Suwon, 443-749, Korea

Abstract

We consider the initial boundary value problem of non-homogeneous stochastic heat equation. The derivative of the solution with respect to time receives heavy random perturbation. The space boundary is Lipschitz and we impose non-zero cylinder condition. We prove a regularity result after finding suitable spaces for the solution and the pre-assigned datum in the problem. The tools from potential theory, harmonic analysis and probability are used. Some Lemmas are as important as the main Theorem.

Keywords: Stochastic heat equation, Lipschitz cylinder domain, Initial-boundary value problem, Anisotropic Besov space.

AMS 2000 subject classifications: primary ; 60H15, secondary; 35R60

1 Introduction

We study the following initial boundary value problem:

$$\begin{cases} du(t,x) = (\Delta u(t,x) + f(t,x))dt + g(t,x)dw_t, & (t,x) \in (0,T) \times D, \\ u(t,x) = b(t,x), & (t,x) \in (0,T) \times \partial D, \\ u(0,x) = u_0, & x \in D, \end{cases} \tag{1.1}$$

where D is a bounded Lipschitz domain in $\mathbb{R}^n$ and $\{w_t(\omega) : t \geq 0, \omega \in \Omega\}$ is a one-dimensional Brownian motion with a probability space Ω. Any solution of (1.1) depends not only (t,x), but also ω. We investigate the regularity of the solution of (1.1) in (t,x) for each ω.

If $g \equiv 0$, the problem is deterministic and the theory has been well-developed. For instance, [5] considered the problem when D is a bounded C^1-domain and [1] and [2] studied the problem when D is a bounded Lipschitz domain. Later, [6] developed a theory using anisotropic Besov spaces. However in our paper, as we let $g \not\equiv 0$, we deal with a stochastic heat equation. This job is nontrivial. Viewing the heat equation in (1.1) as $u_t(t,x) = \Delta u(t,x) + f(t,x) + \dot{w}_t g(t,x)$, we notice that our equation includes an internal source/sink with the white noise coefficient. The (probabilistic) variance of the random noise $\dot{w}_t$, $t \in (0,T)$ is not bounded. Moreover $\dot{w}_{t_1}$ and $\dot{w}_{t_2}$

are independent as long as $t_1 \neq t_2$. Thus, *we do not expect good regularity in time direction since the solution keeps receiving the white noises along the time variable.* An L_p-theory of the Cauchy problem ($D = \mathbb{R}^n$) was established in [12] and since then the initial boundary value problem with zero boundary condition is studied by many authors (see, for instance, [14], [15], [11], [10], [17] and references therein). In this paper we allows the space domain to be Lipschitz and the boundary condition can be non-zero. Moreover, when we do not require the high regularity in x, we consider the joint regularity in (t, x) using anisotropic Besov spaces. The usage of anisotropic Besov spaces is natural with the deterministic heat equation.

Having said that, let us find a formal solution of (1.1); this will be a unique solution in an appropriate space. Firstly, extend u_0 on $\mathbb{R}^n$, f and g on $(0, T) \times \mathbb{R}^n$ (see Section 3 for the mathematical details on these extensions). Let v be a solution of the Cauchy problem, i.e. $D = \mathbb{R}^n$, consisting of (1.1) with the extended u_0 as the initial condition. Let $\hat{h}$ denote the Fourier transform of a function h in $\mathbb{R}^n$. Taking Fourier transform in space on the equation, we have a stochastic differential equation for each frequency $\xi \in \mathbb{R}^n$,

$$d\hat{v}(t,\xi) = (-|\xi|^2 \hat{v}(t,\xi) + \hat{f}(t,\xi)\,)dt + \hat{g}(t,\xi)dw_t.$$

Putting the terms with $\hat{v}$ together in the left hand side, we get

$$d\left(\hat{v}(t,\xi)\, e^{|\xi|^2 t}\right) = e^{|\xi|^2 t}\hat{f}(t,\xi)dt + e^{|\xi|^2 t}\hat{g}(t,\xi)dw_t$$

and hence

$$\hat{v}(t,\xi) = e^{-|\xi|^2 t}\hat{u}_0(\xi) + \int_0^t e^{-|\xi|^2(t-s)}\hat{f}(s,\xi)ds + \int_0^t e^{-|\xi|^2(t-s)}\hat{g}(s,\xi)dw_s.$$

Taking the inverse Fourier transform, we obtain

$$\begin{aligned} v(t,x) = {} & (\Gamma(t,\cdot) *_x u_0)(x) + \textstyle\int_0^t (\,\Gamma(t-s,\cdot) *_x f(s,\cdot)\,)(x)ds \\ & + \textstyle\int_0^t (\,\Gamma(t-s,\cdot) *_x g(s,\cdot)\,)(x)dw_s, \quad t > 0,\ x \in \mathbb{R}^n, \end{aligned} \tag{1.2}$$

where $\Gamma(t,x) := \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} I_{t>0}$ is the inverse Fourier transform of $e^{-|\xi|^2 t} I_{t>0}$ and $*_x$ denotes convolution on x. We restrict v on $\Omega \times (0,T) \times D$. Secondly, we find the solution $h = h(\omega, t, x)$ of the following simple (stochastic) initial-boundary value problem:

$$\begin{cases} h_t(\omega,t,x) = \Delta h(\omega,t,x), \quad (\omega,t,x) \in \Omega \times (0,T) \times D, \\ h(\omega,t,x) = b(\omega,t,x) - v(\omega,t,x), \quad (\omega,t,x) \in \Omega \times (0,T) \times \partial D, \\ h(\omega,0,x) = 0, \quad \omega \in \Omega,\ x \in D. \end{cases} \tag{1.3}$$

Then one can easily check that $u = v + h$ is indeed a solution of (1.1). Since information of h is well known, the estimations of three parts of v in (1.2) are important to us; especially the third one, the *stochastic integral part*.

We are to find a solution space for u and the spaces for f, g, b, u_0 so that the restriction of the three terms in the right hand side of (1.2) on $\Omega \times (0,T) \times D$ and h belong to the solution space and

moreover u is unique in it. We use two types of spaces in this paper; spaces of Bessel potentials and Besov spaces.

In this paper we let $n \geq 2$, $0 < T < \infty$, and D be a bounded Lipschitz domain in $\mathbb{R}^n$. Denote

$$D_T := (0,T) \times D, \quad \partial D_T := (0,T) \times \partial D, \quad \mathbb{R}^n_T := (0,T) \times \mathbb{R}^n.$$

Also, we assume $2 \leq p < \infty$ instead of the usual deterministic setup $1 < p < \infty$; this restriction is due to the stochastic part in (1.1) (see [13]). The main result in this paper is the following.

Theorem 1.1. *Let $2 \leq p < \infty$ and $\frac{1}{p} < k < 1 + \frac{1}{p}$. Assume $f \in \mathbb{B}^{k-2,\frac{1}{2}(k-2)}_{p,o}(D_T)$, $g \in \mathbb{B}^{k-1}_{p,o}(D_T)$, $b \in \mathbb{B}^{k-\frac{1}{p},\frac{1}{2}(k-\frac{1}{p})}_{p}(\partial D_T)$ and $u_0 \in L^p(\Omega, \mathcal{G}_0, \mathcal{U}^{k-\frac{2}{p}}_p(D))$. If $\frac{3}{p} < k < 1 + \frac{1}{p}$, we further assume the compatibility condition $u_0(\omega,x) = b(\omega,0,x)$ for $\omega \in \Omega$, $x \in \partial D$. Then*

(1) if $\frac{1}{p} < k < 1$, there is a unique solution $u \in \mathbb{B}^{k,\frac{1}{2}k}_p(D_T)$ of the initial boundary value problem (1.1) *such that*

$$\begin{aligned} \|u\|_{\mathbb{B}^{k,\frac{1}{2}k}_p(D_T)} \leq c\Big(&\|u_0\|_{L^p(\Omega,\mathcal{G}_0,\mathcal{U}^{k-\frac{2}{p}}_p(D))} + \|f\|_{\mathbb{B}^{k-2,\frac{1}{2}(k-2)}_{p,o}(D_T)} \\ &+\|g\|_{\mathbb{B}^{k-1}_{p,o}(D_T)} + \|b\|_{\mathbb{B}^{k-\frac{1}{p},\frac{1}{2}(k-\frac{1}{p})}_p(\partial D_T)}\Big), \end{aligned} \tag{1.4}$$

where c depends only on D, k, n, p, T.

(2) if $1 \leq k < 1 + \frac{1}{p}$, there is a unique solution $u \in \mathbb{B}^k_p(D_T)$ of the problem (1.1) *such that*

$$\begin{aligned} \|u\|_{\mathbb{B}^{k}_p(D_T)} \leq c\Big(&\|u_0\|_{L^p(\Omega,\mathcal{G}_0,\mathcal{U}^{k-\frac{2}{p}}_p(D))} + \|f\|_{\mathbb{B}^{k-2,\frac{1}{2}(k-2)}_{p,o}(D_T)} \\ &+\|g\|_{\mathbb{B}^{k-1}_{p,o}(D_T)} + \|b\|_{\mathbb{B}^{k-\frac{1}{p},\frac{1}{2}(k-\frac{1}{p})}_p(\partial D_T)}\Big), \end{aligned} \tag{1.5}$$

where c depends only on D, k, n, p, T.

The explanation of spaces and notations appearing in Theorem 1.1 is placed in Section 2.

Remark 1.2. *1. In the part (1) of Theorem 1.1 we estimate the regularity of u in (t,x) simultaneously using anisotropic Besov norm whereas in part (2) we focus on the regularity in x. As we mentioned earlier, the regularity in time is limited while the one in space is not.*

2. If $g \equiv 0$ and $u_0 \equiv 0$, then (1) of Theorem 1.1 coincides with [6].

We organized the paper in the following way. Section 2 explains spaces and notations. In Section 3 we place main lemmas and the proof of Theorem 1.1. The long proofs of some main lemmas are located in Section 4, 5, 6 and 7.

Throughout this paper we denote $A \approx B$ when there are positive constants c_1 and c_2 such that $c_1 A \leq B \leq c_2 A$. Also, $A \lesssim B$ means that there is a positive constant c such that $A \leq cB$. All such constants depend only on n, k, p, T and the Lipschitz constant of ∂D. We use the notations $a \vee b = \max\{a,b\}$, $a \wedge b = \min\{a,b\}$.

2 Preliminaries

Throughout this paper we let $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}, P)$ be a probability space, where $\{\mathcal{G}_t \mid t \geq 0\}$ be a filtration of σ-fields $\mathcal{G}_t \subset \mathcal{G}$ with $\mathcal{G}_0$ containing all P-null subsets of Ω. Assume that a one-dimensional $\{\mathcal{G}_t\}$-adapted Wiener processes $w.$ is defined on $(\Omega, \mathcal{G}, P)$. We denote the mathematical expectation of a random variable $X = X(\omega),\ \omega \in \Omega$ by $E[X]$ or simply EX; we suppress the argument $\omega \in \Omega$ under the expectation E.

For $k \in \mathbb{R}$ let $H_p^k(\mathbb{R}^n)$ be the space of Bessel potential and $B_p^k(\mathbb{R}^n)$ be the Besov space (see, for instance, [3], [20]). For later purpose we place a definition of Besov spaces. Let $\hat{f}(\xi), \xi \in \mathbb{R}^n$ denote the Fourier transform of $f(x), x \in \mathbb{R}^n$ and the space $\mathcal{S}(\mathbb{R}^n)$ denote the *Schwartz space* on $\mathbb{R}^n$. Fix any $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\phi}$ satisfies $\hat{\phi}(\xi) > 0$ on $\frac{1}{2} < |\xi| < 2$, $\hat{\phi}(\xi) = 0$ elsewhere, and $\sum_{j=-\infty}^{\infty} \hat{\phi}(2^{-j}\xi) = 1$ for $\xi \neq 0$. We define ϕ_j and ψ so that their Fourier transforms are given by

$$\begin{array}{ll} \widehat{\phi_j}(\xi) & = \widehat{\phi}(2^{-j}\xi) \quad (j = 0, \pm 1, \pm 2, \cdots) \\ \widehat{\psi}(\xi) & = 1 - \sum_{j=1}^{\infty} \widehat{\phi}(2^{-j}\xi). \end{array} \tag{2.1}$$

Then we define the Besov space $B_p^k(\mathbb{R}^n) = B_{p,p}^k(\mathbb{R}^n)$ by

$$B_p^k(\mathbb{R}^n) = \{ f \in \mathcal{D}(\mathbb{R}^n) \ \mid \ \|f\|_{B_p^k} := \|\psi * f\|_{L^p} + \Big[\sum_{j=1}^{\infty} (2^{kj} \|\phi_j * f\|_{L^p})^p \Big]^{\frac{1}{p}} < \infty \ \},$$

where $\mathcal{D}(\mathbb{R}^n)$ is dual space of Schwartz space and $*$ means the convolution.

2.1 Spaces for D, ∂D and $(0, T)$

When $k \geq 0$, we define

$$H_p^k(D) := \{ F|_D \bigm| F \in H_p^k(\mathbb{R}^n) \}, \quad B_p^k(D) := \{ F|_D \bigm| F \in B_p^k(\mathbb{R}^n) \}, \quad \text{resp.}$$

with the norms

$$\|f\|_{H_p^k(D)} := \inf \|F\|_{H_p^k(\mathbb{R}^n)}, \quad \|f\|_{B_p^k(D)} := \inf \|F\|_{B_p^k(\mathbb{R}^n)}, \quad \text{resp.},$$

where the infima are taken over $F \in H_p^k(\mathbb{R}^n)$ or $F \in B_p^k(\mathbb{R}^n)$ satisfying $F|_D = f$. We also define $B_{p,o}^k(D)$ as the closure of $C_c^\infty(D)$ in $B_p^k(D)$.

Remark 2.1. *Let k_0 be a nonnegative integer. Then the followings hold.*

(1)

$$\|f\|_{H_p^{k_0}(D)}^p \approx \sum_{0 \leq |\beta| \leq k_0} \|D^\beta f\|_{L^p(D)}^p,$$

where $D^\beta = D_{x_1}^{\beta_1} D_{x_2}^{\beta_2} \cdots D_{x_n}^{\beta_n}$ for $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in (\{0\} \cup \mathbb{N})^n$.

(2) *For $k \in (k_0, k_0+1)$*

$$\|f\|^p_{B^k_p(D)} \approx \|f\|^p_{H^{k_0}_p(D)} + \sum_{|\beta|=k_0} \int_D \int_D \frac{|D^\beta f(x) - D^\beta f(y)|^p}{|x-y|^{n+p(k-k_0)}} dxdy.$$

The spaces $B^k_p(\partial D)$, $k \in (0,1)$ are defined similarly.

(3) *Let $k = k_0 + \theta$ with $\theta \in (0,1)$. Then the space $B^k_p(D)$ satisfies the following real interpolation property (see Section 2 of [7]):*

$$(H^{k_0}_p(D), H^{k_0+1}_p(D))_{\theta,p} = B^k_p(D). \tag{2.2}$$

When $k < 0$ we define $B^k_p(D)$ as the dual space of $B^{-k}_{q,o}(D)$ and $B^k_{p,o}(D)$ as the dual space of $B^{-k}_q(D)$, i.e., $B^k_p(D) = (B^{-k}_{q,o}(D))^*$, $B^k_{p,o}(D) = (B^{-k}_q(D))^*$ with $\frac{1}{p} + \frac{1}{q} = 1$.

We define $H^{\frac{1}{2}k}_p(0,T)$, $B^k_p(0,T)$ and $B^k_{p,o}(0,T)$ similarly.

Remark 2.2. *By the subscript o in $B^k_{p,o}(D)$ ($k < 0$) we mean that the natural extension of any distribution in this space vanishes outside D in the following sense. Let $h \in B^k_{p,o}(D) = (B^{-k}_q(D))^*$. We define the extension $\tilde{h} \in B^k_p(\mathbb{R}^n)$ of h by*

$$< \tilde{h}, \Phi > := < h, \Phi|_D >, \quad \Phi \in B^{-k}_q(\mathbb{R}^n);$$

note that by the very definition of $B^{-k}_q(D)$ we have $\Phi|_D \in B^{-k}_q(D)$ and $< h, \Phi|_D >$ is well defined; here the condition that D is Lipschitz is used. Then for any Φ with its support outside D, then $< \tilde{h}, \Phi > = 0$. This means that $\tilde{h}$ vanishes outside D. A similar reasoning says that the extension of any distribution in $B^k_p(D)$ may not vanish outside D and hence we do not add the subscript o.

For the initial condition u_0 we need

$$\mathcal{U}^k_p(D) := \begin{cases} B^k_p(D), & k \geq 0, \\ B^k_{p,o}(D), & k < 0. \end{cases} \tag{2.3}$$

2.2 Spaces for D_T, ∂D_T

For $k \geq 0$ we define the anisotropic Besov space $B^{k,\frac{1}{2}k}_p(D_T)$ by

$$B^{k,\frac{1}{2}k}_p(D_T) := L^p\Big((0,T); B^k_p(D)\Big) \cap L^p\left(D; B^{\frac{k}{2}}_p((0,T))\right) \tag{2.4}$$

with the norm

$$\|f\|_{B^{k,\frac{1}{2}k}_p(D_T)} := \Big(\int_0^T \|f(t,\cdot)\|^p_{B^k_p(D)} dt\Big)^{\frac{1}{p}} + \Big(\int_D \|f(\cdot,x)\|^p_{B^{\frac{k}{2}}_p((0,T))} dx\Big)^{\frac{1}{p}}, \tag{2.5}$$

where $B^{\frac{k}{2}}_p((0,T))$ is defined similarly as in Section 2.1; we also define

$$B^{k,\frac{1}{2}k}_{p,o}(D_T) = L^p\Big((0,T); B^k_{p,o}(D)\Big) \cap L^p\Big(D; B^{\frac{k}{2}}_{p,o}((0,T))\Big)$$

with the same norm (2.5).

For $k < 0$ we define $B^{k,\frac{1}{2}k}_p(D_T) = (B^{-k,-\frac{1}{2}k}_{q,o}(D_T))^*$ and $B^{k,\frac{1}{2}k}_{p,o}(D_T) = (B^{-k,-\frac{1}{2}k}_q(D_T))^*$ with $\frac{1}{p} + \frac{1}{q} = 1$.

We define $B^{k,\frac{1}{2}k}_p(\partial D_T)$, $B^{k,\frac{1}{2}k}_{p,o}(\partial D_T)$, $k \in (0,1)$ similarly.

2.3 Stochastic Banach spaces

The solution u and functions f, g, b, u_0 in (1.1) are all random. Using Section 2.1 and 2.2 we construct the spaces for them. We describe two types of spaces. The first type emphasizes the regularity in x whereas the second type does the regularity in t, x together. Again, let $k \in \mathbb{R}$.

We can consider u, f, g, b as function space-valued stochastic processes and hence $(\Omega \times (0,T), \mathcal{P}, P \bigotimes \ell((0,T]))$ is a suitable choice for their common domain, where $\mathcal{P}$ is the predictable σ-field generated by $\{\mathcal{G}_t : t \geq 0\}$ (see, for instance, pp. 84–85 of [12]) and $\ell((0,T])$ is the Lebesgue measure on $(0,T)$. We define

$$\mathbb{H}_p^k(\mathbb{R}_T^n) = L^p(\Omega \times (0,T), \mathcal{P}, H_p^k(\mathbb{R}^n)), \quad \mathbb{B}_p^k(\mathbb{R}_T^n) = L^p(\Omega \times (0,T), \mathcal{P}, B_p^k(\mathbb{R}^n))$$

and the norms

$$\|f\|_{\mathbb{H}_p^k(\mathbb{R}_T^n)} = \left(E\int_0^T \|f(s,\cdot)\|_{H_p^k(\mathbb{R}^n)}^p ds\right)^{\frac{1}{p}}, \quad \|f\|_{\mathbb{B}_p^k(\mathbb{R}_T^n)} = \left(E\int_0^T \|f(s,\cdot)\|_{B_p^k(\mathbb{R}^n)}^p ds\right)^{\frac{1}{p}}$$

; we suppress ω in f. Similarly we define

$$\mathbb{H}_p^k(D_T) = L^p(\Omega \times (0,T], \mathcal{P}, H_p^k(D)), \quad \mathbb{B}_p^k(D_T) = L^p(\Omega \times (0,T), \mathcal{P}, B_p^k(D)),$$

$$\mathbb{B}_{p,o}^k(D_T) = L^p(\Omega \times (0,T), \mathcal{P}, B_{p,o}^k(D)).$$

We also define the stochastic *anisotropic Besov* spaces

$$\mathbb{B}_p^{k,\frac{1}{2}k}(D_T) = L^p(\Omega, \mathcal{G}, B_p^{k,\frac{1}{2}k}(D_T)), \quad \mathbb{B}_p^{k,\frac{1}{2}k}(\partial D_T) = L^p(\Omega, \mathcal{G}, B_p^{k,\frac{1}{2}k}(\partial D_T))$$

with norms

$$\|f\|_{\mathbb{B}_p^{k,\frac{1}{2}k}(D_T)} = \left(E\|f\|_{B_p^{k,\frac{1}{2}k}(D_T)}^p\right)^{\frac{1}{p}}, \quad \|f\|_{\mathbb{B}_p^{k,\frac{1}{2}k}(\partial D_T)} = \left(E\|f\|_{B_p^{k,\frac{1}{2}k}(\partial D_T)}^p\right)^{\frac{1}{p}}.$$

Similarly we define $\mathbb{B}_{p,o}^{k,\frac{1}{2}k}(D_T) = L^p(\Omega, \mathcal{G}, B_{p,o}^{k,\frac{1}{2}k}(D_T))$.

3 Lemmas and Proof of Theorem 1.1

In this section we estimate the three terms of (1.2) and prove our main theorem.

For $l < 0$, if $h \in B_{p,o}^l(D) = (B_q^{-l}(D))^*$, then we define $\tilde{h} \in B_p^l(\mathbb{R}^n)$ as the trivial extension of h by

$$< \tilde{h}, \phi > := < h, \phi|_D >, \quad \phi \in B_q^{-l}(\mathbb{R}^n), \tag{3.1}$$

; note $\|\tilde{h}\|_{B_p^l(\mathbb{R}^n)} \approx \|h\|_{B_{p,o}^l(D)}$. For $l \geq 0$, if $h \in B_p^l(D)$, then we define $\tilde{h} \in B_p^l(\mathbb{R}^n)$ as the Stein's extension of h with $\|\tilde{h}\|_{B_p^l(\mathbb{R}^n)} \lesssim \|h\|_{B_p^l(D)}$ (see section 2 of [7] and Chapter 6 of [19]); this extension is possible since our space domain D is at least Lipschitz. Recall the definition of $\mathcal{U}_p^l(D)$ in (2.3).

Lemma 3.1. *Let $0 < k < 2$. We assume $u_0(\omega,\cdot) \in \mathcal{U}_p^{k-\frac{2}{p}}(D)$ for each $\omega \in \Omega$. Let $\tilde{u}_0$ denote the extension of u_0 (trivial or Stein's). For each $(\omega,t,x) \in \Omega \times (0,T) \times \mathbb{R}^n$ define*

$$v_1(\omega,t,x) := \begin{cases} <\tilde{u}_0(\omega,\cdot), \Gamma(t,x-\cdot)>, & \text{if } 0 \le k < \frac{2}{p}, \\ \int_{\mathbb{R}^n} \Gamma(t,x-y)\tilde{u}_0(\omega,y)\,dy, & \text{if } \frac{2}{p} \le k. \end{cases} \tag{3.2}$$

Then $v_1(\omega,\cdot,\cdot) \in B_p^{k,\frac{1}{2}k}(\mathbb{R}^n_T)$ for each ω and

$$\|v_1(\omega,\cdot,\cdot)\|_{B_p^{k,\frac{1}{2}k}(\mathbb{R}^n_T)} \le c\,\|u_0(\omega,\cdot)\|_{\mathcal{U}_p^{k-\frac{2}{p}}(D)}, \quad \omega \in \Omega, \tag{3.3}$$

where c is independent of u_0 and ω.

; the proof is presented in Section 4.

For $0 < k < 2$ and $h = h(t,x) \in B_{p,o}^{k-2,\frac{1}{2}k-1}(D_T)$ we define $\tilde{h} \in B_p^{k-2,\frac{1}{2}k-1}(\mathbb{R}^{n+1})$ by

$$<\tilde{h},\phi>:=<h,\phi|_{D_T}>, \quad \phi \in B_q^{2-k,1-\frac{1}{2}k}(\mathbb{R}^{n+1}). \tag{3.4}$$

In this case $\|\tilde{h}\|_{B_p^{k-2,\frac{1}{2}k-1}(\mathbb{R}^{n+1})} \approx \|h\|_{B_{p,o}^{k-2,\frac{1}{2}k-1}(D_T)}$.

Lemma 3.2. *Let $0 < k < 2$ and $f \in \mathbb{B}_{p,o}^{k-2,\frac{1}{2}k-1}(D_T)$. Define*

$$v_2(\omega,t,x) :=< \tilde{f}(\omega,\cdot,\cdot), \Gamma(t-\cdot,x-\cdot)> . \tag{3.5}$$

Then $v_2 \in \mathbb{B}_p^{k,\frac{1}{2}k}(\mathbb{R}^n_T)$ and

$$\|v_2\|_{\mathbb{B}_p^{k,\frac{1}{2}k}(\mathbb{R}^n_T)} \le c\|f\|_{\mathbb{B}_{p,o}^{k-2,\frac{1}{2}k-1}(D_T)}. \tag{3.6}$$

; the proof is in Section 5.

Before we estimate v_3 let us place the following lemma which is Exercise 5.8.6 in [3]:

Lemma 3.3. *Assume that A_0 and A_1 are Banach spaces and that $1 \le p < \infty$, $0 < \theta < 1$. Then*

$$(L_p(A_0), L_p(A_1))_{\theta,p} = L_p((A_0,A_1)_{\theta,p}),$$

where $(\cdot,\cdot)_{\theta,p}$ is a real interpolation.

If $0 < k < 1$, then for $g = g(\omega,t,x) \in \mathbb{B}_{p,o}^{k-1}(D_T)$ we define $\tilde{g} \in \mathbb{B}_p^{k-1}(\mathbb{R}^{n+1})$ by

$$<\tilde{g}(\omega,t,\cdot),\phi>:=<g(\omega,t,\cdot),\phi|_{D_T}>, \quad \phi \in B_q^{k-1}(\mathbb{R}^n) \tag{3.7}$$

and, if $k \ge 1$, we define $\tilde{g}(\omega,t,\cdot) \in B_p^{k-1}(\mathbb{R}^{n+1})$ by $\tilde{g}(\omega,t,x) = g(\omega,t,x)$ for $x \in D$ and $\tilde{g}(\omega,t,x) = 0$ for $x \in \mathbb{R}^n \setminus \bar{D}$. Then we get $\|\tilde{g}\|_{\mathbb{B}_p^{k-1}(\mathbb{R}^{n+1})} \approx \|g\|_{\tilde{\mathbb{B}}_p^{k-1}(D_T)}$.

Lemma 3.4. *Let $k > 0$ and $g \in \mathbb{B}^{k-1}_{p,o}(D_T)$. Define*

$$v_3(t,x) := \begin{cases} \int_0^t < \tilde{g}(s,\cdot), \Gamma(t-s, x-\cdot) > dw_s, & \text{if} \quad 0 < k < 1, \\ \\ \int_0^t \int_{\mathbb{R}^n} \Gamma(t-s, x-y)\tilde{g}(s,y)dy\, dw_s, & \text{if} \quad 1 \le k \end{cases} \tag{3.8}$$

;we suppressed ω. Then $v_3 \in \mathbb{B}^k_p(\mathbb{R}^n_T)$ with

$$\|v_3\|_{\mathbb{B}^k_p(\mathbb{R}^n_T)} \le c\|g\|_{\mathbb{B}^{k-1}_{p,o}(D_T)}. \tag{3.9}$$

Proof. Apply the result in [12] and Lemma 3.3. □

For $\epsilon \in (0,1)$ we let $p_0 = \frac{1}{2} + \frac{1}{2}\epsilon,\ p_0^{'} = \frac{1}{2} - \frac{1}{2}\epsilon$. We say that $(\frac{1}{p}, k) \in \mathcal{R}_\epsilon$ if α and p are numbers satisfying one of the followings:

1. $p_0 < p < p_0^{'}$ if $0 < k < 1$,
2. $1 < p \le p_0$ if $\frac{2}{p} - 1 - \epsilon < k < 1$,
3. $p_0^{'} \le p < \infty$ if $0 < k < \frac{2}{p} + \epsilon$.

Lemma 3.5. *There is a positive constant $\epsilon \in (0,1)$ depending only on Lipschitz constant of ∂D such that if $(\frac{1}{p}, k) \in \mathcal{R}_\epsilon$, then for all $b' \in \mathbb{B}^{k,\frac{1}{2}k}_p(\partial D_T)$ with $b'(\omega, 0, x) = 0$ for $\omega \in \Omega, x \in \partial D$ if $k > \frac{2}{p}$. Then there is a unique solution $h \in \mathbb{B}^{k+\frac{1}{p},\frac{1}{2}k+\frac{1}{2p}}_p(D_T)$ of the problem (1.3) in $\Omega \times D_T$ with boundary value b' in place of $b - v$ and $h(\omega, 0, x) = 0$ for $\omega \in \Omega,\ x \in D$ and it satisfies*

$$\|h\|_{\mathbb{B}^{k+\frac{1}{p},\frac{1}{2}k+\frac{1}{2p}}_p(D_T)} \le c\|b'\|_{\mathbb{B}^{k,\frac{1}{2}k}_p(\partial D_T)}. \tag{3.10}$$

If D is a C^1-domain, then we can take $\epsilon = 1$.

Proof. Apply [1], [2] and [6] for each $\omega \in \Omega$. □

We need the following restriction theorem from [4]:

Lemma 3.6. *Let $\frac{1}{p} < k < 1 + \frac{1}{p}$. Then for any $h = h(t,x) \in B^{k,\frac{1}{2}k}_p(\mathbb{R}^n_T)$, we have $h|_{\partial D_T} \in B^{k-\frac{1}{p},\frac{1}{2}k-\frac{1}{2p}}_p(\partial D_T)$.*

The following lemma for the stochastic part v_3 in (1.2) is important and we elaborate the proof in Section 6 and 7.

Lemma 3.7. *Assume $2 \le p < \infty$.*

(1) Let $\frac{1}{p} < k < 1$ and $g \in \mathbb{B}^{k-1}_{p,o}(D_T)$. Then v_3 defined for such k in Lemma 3.4 belongs to $\mathbb{B}^{k,\frac{1}{2}k}_p(\mathbb{R}^n_T)$ and

$$\|v_3\|_{\mathbb{B}^{k,\frac{1}{2}k}_p(\mathbb{R}^n_T)} \le c\|g\|_{\mathbb{B}^{k-1}_{p,o}(D_T)}. \tag{3.11}$$

(2) Let $1 \leq k < 1+\frac{1}{p}$ and $g \in \mathbb{B}^{k-1}_{p,o}(D_T)$. Then v_3 defined for such k in Lemma 3.4 satisfies

$$\|v_3|_{\partial D_T}\|_{\mathbb{B}_p^{k-\frac{1}{p},\frac{1}{2}k-\frac{1}{2p}}(\partial D_T)} \leq c\|g\|_{\mathbb{B}^{k-1}_{p,o}(D_T)} \tag{3.12}$$

By Lemma 3.1 - Lemma 3.7 the proof of Theorem 1.1 follows.

Proof of Theorem 1.1 Recall the derivation of the solution $u = v + h$ in Section 1.

(1) By Lemma 3.1, 3.2 and Lemma 3.7 (1), the (random) function $v := v_1 + v_2 + v_3$ is in $\mathbb{B}_p^{k,\frac{1}{2}k}(\mathbb{R}^n_T)$; note that the definition of u_1 in Lemma 3.1 is different by the cases $k \in (\frac{1}{p}, \frac{2}{p})$ and $k \in [\frac{2}{p}, 1)$. Moreover, we choose the definition of u_3 in Lemma 3.4 for $k \in (\frac{1}{p}, 1)$. Now, using Lemma 3.6 for each $\omega \in \Omega$, we have $b' := b - v|_{\partial D_T} \in \mathbb{B}_p^{k-\frac{1}{p},\frac{1}{2}k-\frac{1}{2p}}(\partial D_T)$. Let $u_4 \in \mathbb{B}_p^{k,\frac{1}{2}k}(D_T)$ be the unique solution of the problem (1.3) which does exist by Lemma 3.5. Then $u := v + h$ is a solution of (1.1) and the estimate (1.4) follows (3.3), (3.6), (3.11) and (3.10). The uniqueness of such u follows the theory of deterministic heat equation.

(2) Set v as in (1) by choosing the appropriate definitions of v_1, v_3 when $k \in [1, 1+\frac{1}{p})$. Then proof is similar to the case (1). However, this time we can not have v_3 in $\mathbb{B}_p^{k,\frac{1}{2}k}(\mathbb{R}^n_T)$ although it is in $\mathbb{B}_p^{k}(\mathbb{R}^n_T)$ by the Lemma 3.4. Hence, we have v is in $\mathbb{B}_p^{k}(\mathbb{R}^n_T)$ as v_1, v_2 are trivially in $\mathbb{B}_p^{k}(\mathbb{R}^n_T)$ (see (2.4)). Nevertheless, by using Lemma 3.7 (2) we still have $b' \in \mathbb{B}_p^{k-\frac{1}{p},\frac{1}{2}k-\frac{1}{2p}}(\partial D_T)$. By choosing v_4 as before in $\mathbb{B}_p^{k,\frac{1}{2}k}(D_T)$ and hence $\mathbb{B}_p^{k}(\mathbb{R}^n_T)$, we have a solution of (1.1) in $\mathbb{B}_p^{k}(\mathbb{R}^n_T)$ and the estimate (1.5) follows (3.3), (3.6), (3.12), (3.10) with (2.4). The solution is unique. $\square$

4 Proof of Lemma 3.1

We believe that one may find a proof of Lemma 3.1 is in the literature. However, we can not find the exact reference and, hence, we provide our own proof. We start with a lemma for multipliers.

Lemma 4.1. *Let $\Phi(\xi) = \hat{\phi}(2^{-1}\xi) + \hat{\phi}(\xi) + \hat{\phi}(2\xi)$ with ϕ in the definition of Besov spaces, $\Phi_j(\xi) = \Phi(2^{-j}\xi)$, and $\rho_{tj}(\xi) = \Phi_j(\xi)e^{-t|\xi|^2}$ for each integer j. Then $\rho_{tj}(\xi)$ is a $L^p(\mathbb{R}^n)$-multiplier with the finite norm $M(t,j)$ for $1 < p < \infty$. Moreover for $t > 0$*

$$M(t,j) \lesssim e^{-\frac{1}{4}t2^{2j}} \sum_{0\leq i\leq n} t^i 2^{2ij} \lesssim e^{-\frac{1}{8}t2^{2j}}. \tag{4.1}$$

Proof. The $L^p(\mathbb{R}^n)$-multiplier norm $M(t,j)$ of $\rho_{tj}(\xi)$ is equal to the $L^p(\mathbb{R}^n)$-multiplier norm of $\rho'_{tj}(\xi) := \Phi(\xi)e^{-t2^{2j}|\xi|^2}$ (see Theorem 6.1.3 in [3]). Now, we make use of the Theorem 4.6´ of [19]. We assume $\beta_1, \beta_2, \cdots, \beta_l = 1$ and $\beta_i = 0$ for $l+1 \leq i \leq n$, and set $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$. Since *supp* $(\Phi) \subset \{\xi \in \mathbb{R}^n \mid \frac{1}{4} < |\xi| < 4\}$, we have

$$|D_\xi^\beta \rho'_{tj}(\xi)| \lesssim \sum_{0\leq i\leq|\beta|} t^i 2^{2ij} e^{-\frac{1}{4}t2^{2j}} \chi_{\frac{1}{4}<|\xi|<4}(\xi),$$

where χ_A is the characteristic function on a set A. Hence, for $A = \prod_{1\leq i\leq l}[2^{k_i}, 2^{k_i+1}]$ we receive

$$\int_A \left|\frac{\partial^{|\beta|}}{\partial\xi_\beta}\rho'_{tj}(\xi)\right| d\xi_\beta \leq c \sum_{0\leq i\leq n} t^i 2^{2ij} e^{-\frac{1}{4}t2^{2j}}.$$

$\square$

Below $\tilde{u}_0$ is the extension of u_0; note $\tilde{u}_0(\omega,\cdot) \in B_p^{k-\frac{2}{p}}(\mathbb{R}^n)$ for each $\omega \in \Omega$. The following lemma handles the case $k=0$.

Lemma 4.2. *We have*

$$\|v_1(\omega)\|_{L^p(\mathbb{R}^n_T)} \le c\, \|\tilde{u}_0(\omega)\|_{B_p^{-\frac{2}{p}}(\mathbb{R}^n)}, \quad \omega \in \Omega, \tag{4.2}$$

where the constant c is independent of u_0 and ω.

Proof. We may assume that $\tilde{u}_0 \in C_0^\infty(\mathbb{R}^n)$ since $C_0^\infty(\mathbb{R}^n)$ is dense in $B_p^{-\frac{2}{p}}(\mathbb{R}^n)$. We use the dyadic partition of unity $\hat{\psi}(\xi) + \sum_{j=1}^\infty \hat{\phi}(2^{-j}\xi) = 1$ for $\xi \in \mathbb{R}^n$, so that we can write

$$\hat{v}_1(t,\xi) = \hat{\psi}(\xi) e^{-t|\xi|^2} \widehat{\tilde{u}_0}(\xi) + \sum_{j=1}^\infty \hat{\phi}(2^{-j}\xi) e^{-t|\xi|^2} \widehat{\tilde{u}_0}(\xi).$$

For $t>0$ we have

$$\begin{aligned}&\textstyle\int_{\mathbb{R}^n} |v_1(t,x)|^p dx \\ &\le \textstyle\int_{\mathbb{R}^n} \left| \mathcal{F}^{-1}\left(e^{-t|\xi|^2} \hat{\psi}(\xi)\, \widehat{\tilde{u}_0}(\xi)\right)(x)\right|^p dx + \int_{\mathbb{R}^n} \left|\mathcal{F}^{-1}\Big(\sum_{j=1}^\infty e^{-t|\xi|^2} \hat{\phi}_j(\xi)\, \widehat{\tilde{u}_0}(\xi)\Big)(x)\right|^p dx.\end{aligned} \tag{4.3}$$

The first term on the right-hand side of (4.3) is dominated by

$$\|\psi * \tilde{u}_0\|^p_{L^p(\mathbb{R}^n)}. \tag{4.4}$$

Now, we estimate the second term on the right-hand side of (4.3). We use the facts that $\hat{\phi}_j = \Phi_j \hat{\phi}_j$ for all j, where Φ_j is defined in Lemma 4.1. By Lemma 4.1, $\Phi_j(\xi) e^{-t|\xi|^2}s$ are the $L^p(\mathbb{R}^n)$-Fourier multipliers with the norms $M(t,j)$. Then we divide the sum as

$$\begin{aligned}&\int_{\mathbb{R}^n} \left|\mathcal{F}^{-1}\Big(\sum_{j=1}^\infty e^{-t|\xi|^2} \hat{\phi}_j(\xi)\, \widehat{\tilde{u}_0}(\xi)\Big)(x)\right|^p dx \\ &= \int_{\mathbb{R}^n} \left|\mathcal{F}^{-1}\Big(\sum_{j=1}^\infty \Phi_j(\xi) e^{-t|\xi|^2} \hat{\phi}_j(\xi)\, \widehat{\tilde{u}_0}(\xi)\Big)(x)\right|^p dx \\ &\le \Big(\sum_{2^{2j} \le 1/t} M(t,j) \|\tilde{u}_0 * \phi_j\|_{L^p}\Big)^p + \Big(\sum_{2^{2j} \ge 1/t} M(t,j) \|\tilde{u}_0 * \phi_j\|_{L^p}\Big)^p \\ &=: I_1(t) + I_2(t).\end{aligned}$$

By Lemma 4.1 we have $M(t,j) \le c$ for $t2^{2j} \le 1$. We take a satisfying $-\frac{2}{p} < a < 0$ and then use

Hölder inequality to get

$$\begin{aligned}
\int_0^T I_1(t)^p dt &\lesssim \int_0^T \Big(\sum_{2^{2j}\le 1/t} 2^{-\frac{p}{p-1}aj}\Big)^{p-1} \sum_{2^{2j}\le 1/t} 2^{paj}\|\phi_j * \tilde u_0\|_{L^p}^p dt \\
&\lesssim \int_0^T t^{\frac12 pa} \sum_{2^{2j}\le 1/t} 2^{paj}\|\phi_j * \tilde u_0\|_{L^p}^p dt \\
&\lesssim \sum_{j=1}^\infty 2^{paj}\|\phi_j * \tilde u_0\|_{L^p}^p \int_0^{2^{-2j}} t^{\frac12 pa} dt \\
&= c\sum_{j=1}^\infty 2^{-2j}\|\phi_j * \tilde u_0\|_{L^p}^p.
\end{aligned}$$

By Lemma 4.1 again $M(t,j)\le c(t2^{2j})^{-m}\sum_{0\le i\le n}(t2^{2j})^i \le c2^{(2n-2m)j}t^{n-m}$ for $t\cdot 2^{2j}\ge 1$ and $m>0$. We fix $b>0$ and then choose m satisfying $p\,2(n-m)+\frac12 p\,b+1<0$, so that we obtain

$$\begin{aligned}
\int_0^T I_2(t)^p dt &\lesssim \int_0^T \Big(\sum_{2^{2j}\ge 1/t} 2^{(2n-2m)j} t^{n-m}\|\phi_j * \tilde u_0\|_{L^p}\Big)^p dt \\
&\lesssim \int_0^\infty t^{p(n-m)}\Big(\sum_{2^{2j}\ge 1/t} 2^{-\frac{p}{p-1}bj}\Big)^{p-1} \sum_{2^{2j}\ge 1/t} 2^{pbj}2^{p(2n-2m)j}\|\phi_j * \tilde u_0\|_{L^p}^p dt \\
&\lesssim \int_0^\infty t^{p(n-m)+\frac12 pb} \sum_{2^{2j}\ge 1/t} 2^{pbj}2^{p(2n-2m)j}\|\phi_j * \tilde u_0\|_{L^p}^p dt \\
&\lesssim \sum_{j=1}^\infty 2^{pbj}2^{p(2n-2m)j}\|\phi_j * \tilde u_0\|_{L^p}^p \int_{2^{-2j}}^\infty t^{p(n-m)+\frac12 pb} dt \\
&= c\sum_{j=1}^\infty 2^{-2j}\|\phi_j * \tilde u_0\|_{L^p}^p.
\end{aligned}$$

□

Proof of Lemma 3.1 The following is a classical result (see [16]):

$$\int_0^T \|v_1(\omega,t,\cdot)\|_{H_p^2(\mathbb{R}^n)}^p dt + \int_{\mathbb{R}^n} \|v_1(\omega,\cdot,x)\|_{H_p^1(0,T)}^p dx \le c\|\tilde u_0(\omega)\|^p_{B_p^{2-\frac2p}(\mathbb{R}^n)}, \quad \omega\in\Omega. \tag{4.5}$$

Using (4.5), Lemma 4.2 and the following real interpolations

$$\begin{aligned}
(L^p(\mathbb{R}^n), H_p^2(\mathbb{R}^n))_{\frac k2,p} = B_p^k(\mathbb{R}^n), &\quad (L^p((0,T)), H_p^1((0,T))_{\frac k2,p} = B_p^{\frac k2}((0,T)), \\
(B_p^{-\frac2p}(\mathbb{R}^n), B_p^{2-\frac2p}(\mathbb{R}^n))_{\frac k2,p} &\quad = \quad B_p^{k-\frac2p}(\mathbb{R}^n),
\end{aligned}$$

we have

$$\|v_1(\omega)\|^p_{B^{k,\frac12 k}(\mathbb{R}^n_T)} = \int_0^T \|v_1(\omega,t,\cdot)\|^p_{B_p^k(\mathbb{R}^n)}dt + \int_{\mathbb{R}^n} \|v_1(\omega,\cdot,x)\|^p_{B_p^{\frac k2}(0,T)}dx \le c\|\tilde u_0(\omega)\|^p_{B_p^{k-\frac2p}(\mathbb{R}^n)}.$$

This implies Lemma 3.1. □

5 Proof of Lemma 3.2

We need the space of the *parabolic* Bessel potentials. For $l \in \mathbb{R}$ the parabolic Bessel potential Π_l is a distribution whose Fourier transform in $\mathbb{R}^{n+1}$ is defined by

$$\widehat{\Pi_l}(\tau,\xi) = c_k(1 + i\tau + |\xi|^2)^{-\frac{l}{2}}, \quad \tau \in \mathbb{R},\, \xi \in \mathbb{R}^n.$$

In particular, if $l > 0$, then

$$\Pi_l(t,x) = \begin{cases} c_l\, t^{\frac{l-n-2}{2}}\, e^{-t}\, e^{-\frac{|x|^2}{4t}} & \text{if } t > 0, \\ 0 & \text{if } t \le 0 \end{cases} \tag{5.1}$$

; see [8]. In particular, $\Pi_2 = e^{-t}\Gamma$, where Γ is the heat kernel introduced in Section 1.

For $1 \le p < \infty$ we define the space of the parabolic Bessel potentials, $H_p^{l,\frac{1}{2}l}(\mathbb{R}^{n+1})$, by

$$H_p^{l,\frac{1}{2}l}(\mathbb{R}^{n+1}) = \{f \in \mathcal{S}'(\mathbb{R}^{n+1}) \mid \Pi_{-l} * f \in L^p(\mathbb{R}^{n+1})\}$$

with the norm

$$\|f\|_{H_p^{l,\frac{1}{2}l}(\mathbb{R}^{n+1})} = \|\Pi_{-l} * f\|_{L^p(\mathbb{R}^{n+1})},$$

where $*$ in this case is a convolution in $\mathbb{R}^{n+1}$ and $\mathcal{S}'(\mathbb{R}^{n+1})$ is the dual space of the Schwartz space $\mathcal{S}(\mathbb{R}^{n+1})$. Note that if $l \ge 0$, we have

$$H_p^{l,\frac{1}{2}l}(\mathbb{R}^{n+1}) = L^p\Big(\mathbb{R}; H_p^l(\mathbb{R}^n)\Big) \cap L^p\left(\mathbb{R}^n; H_p^{\frac{1}{2}l}(\mathbb{R})\right).$$

For $l \ge 0$ we define

$$H_p^{l,\frac{1}{2}l}(\mathbb{R}^n_T) := \{f|_{\mathbb{R}^n_T} \ \mid\ f \in H_p^{l,\frac{1}{2}l}(\mathbb{R}^{n+1})\}$$

and let $H_{p,o}^{l,\frac{1}{2}l}(\mathbb{R}^n_T)$ be the closure of $C_c^\infty(\mathbb{R}^n_T)$ in $H_p^{l,\frac{1}{2}l}(\mathbb{R}^n_T)$.

For $l < 0$ we also define $H_p^{l,\frac{1}{2}l}(\mathbb{R}^n_T)$ and $H_{p,o}^{l,\frac{1}{2}l}(\mathbb{R}^n_T)$ as the dual spaces of $H_{q,o}^{-l,-\frac{1}{2}l}(\mathbb{R}^n_T)$ and $H_q^{-l,-\frac{1}{2}l}(\mathbb{R}^n_T)$ respectively with $\frac{1}{p}+\frac{1}{q} = 1$; $H_p^{l,\frac{1}{2}l}(\mathbb{R}^n_T) = (H_{q,o}^{-l,-\frac{1}{2}l}(\mathbb{R}^n_T))^*$, $H_{p,o}^{l,\frac{1}{2}l}(\mathbb{R}^n_T) = (H_q^{-l,-\frac{1}{2}l}(\mathbb{R}^n_T))^*$.

Proof of Lemma 3.2 We assumed $0 < k < 2$ and $f \in \mathbb{B}_{p,o}^{k-2,\frac{1}{2}k-1}(D_T)$. Let $\tilde{f}$ is the extension of f on $\mathbb{R}^{n+1}$.

1. We just show the case $k = 0$

$$\|u_2(\omega)\|_{L^p(\mathbb{R}^n_T)} \le c\|\tilde{f}(\omega)\|_{H_p^{-2,-1}(\mathbb{R}^{n+1})}, \quad \omega \in \Omega. \tag{5.2}$$

Then the classical result ([16]):

$$\|u_2(\omega)\|_{H_p^{2,1}(\mathbb{R}^n_T)} \le c\|\tilde{f}(\omega)\|_{L^p(\mathbb{R}^{n+1})}, \quad \omega \in \Omega$$

and the real interpolations

$$(L^p(\mathbb{R}^n_T), H_p^{2,1}(\mathbb{R}^n_T))_{\frac{k}{2},p} = B_p^{k,\frac{1}{2}k}(\mathbb{R}^n_T), \quad (H_p^{-2,-1}(\mathbb{R}^{n+1}), L^p(\mathbb{R}^{n+1}))_{\frac{k}{2},p} = B_p^{k-2,\frac{1}{2}k-1}(\mathbb{R}^{n+1})$$

lead us to

$$\|u_2(\omega)\|_{B_p^{k,\frac{1}{2}k}(\mathbb{R}^n_T)} \le c\|\tilde f(\omega)\|_{B_p^{k-2,\frac{1}{2}k-1}(\mathbb{R}^{n+1})}, \quad \omega\in\Omega \tag{5.3}$$

and (3.6) follows.

2. Since $C_c^\infty(\mathbb{R}^n_T)$ is dense in $H^{l,\frac{1}{2}l}_{p,o}(\mathbb{R}^n_T)$ even for $l<0$, we may assume $\tilde f$ is in $C_c^\infty(\mathbb{R}^n_T)$. In this case the representation

$$u_2(\omega,t,x)=\int_0^t\int_{\mathbb{R}^n}\Gamma(t-s,x-y)\tilde f(\omega,s,y)\,dy\,ds$$

is legal. Recalling $\Pi_2(t,x)=e^{-t}\Gamma(t,x)$, we have

$$u_2(\omega,t,x)=\int_0^t\int_{\mathbb{R}^n}e^{t-s}\Pi_2(t-s,x-y)\tilde f(\omega,s,y)\,dy\,ds=e^t(\Pi_2*g(\omega,t,x)),$$

where $g(\omega,s,y)=e^{-s}\tilde f(\omega,s,y)$. Hence,

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n}|u_2(\omega,t,x)|^p dxdt &= \int_0^T\int_{\mathbb{R}^n}e^{pt}|\Pi_2*g(\omega,t,x)|^p dxdt\\ &\le e^{pT}\int_0^\infty\int_{\mathbb{R}^n}|\Pi_2*g(\omega,t,x)|^p dxdt\\ &\le e^{pT}\|g(\omega)\|^p_{H_p^{-2,-1}(\mathbb{R}^{n+1})}\\ &\lesssim e^{pT}\|\tilde f(\omega)\|^p_{H_p^{-2,-1}(\mathbb{R}^{n+1})},\end{aligned}$$

where the last inequality follows by

$$|<g(\omega),\phi>|=|<\tilde f(\omega),e^{-t}\phi>|\le\|\tilde f(\omega)\|_{H_p^{-2,-1}(\mathbb{R}^{n+1})}\,\|e^{-t}\phi\|_{H_p^{2,1}(\mathbb{R}^{n+1})},\quad \phi\in H_p^{2,1}(\mathbb{R}^{n+1})$$

and the fact $\|e^{-t}\phi\|_{H_p^{2,1}(\mathbb{R}^{n+1})}\lesssim\|\phi\|_{H_p^{2,1}(\mathbb{R}^{n+1})}$. We have received (5.2) and the lemma is proved. □

6 Proof of Lemma 3.7 (1)

We only need to prove the case $T=1$:

$$E\int_{\mathbb{R}^n}\int_0^1\int_0^1\frac{|v_3(t,x)-v_3(s,x)|^p}{|t-s|^{1+\frac{p}{2}k}}dsdt\,dx\lesssim\|\tilde g\|^p_{L^p(\Omega\times(0,1),\mathcal{P},B_p^{k-1}(\mathbb{R}^n))}, \tag{6.1}$$

where $\tilde g$ is the extension of g and v_3 is defined in (3.8) using $\tilde g$. Then the general case follows a scaling argument with the fact that under the expectation we can use any Brownian motion in the definition of v_3 and the observation that $\bar w_r:=\frac{1}{\sqrt{T}}w_{\sqrt{T}r},\ r\in[0,1]$ is also a Brownian motion. Indeed, let $\tilde g(\omega,r,y),\ \omega\in\Omega,\ r\in[0,T],\ y\in\mathbb{R}^n$ be given. Notice that we may assume that $\tilde g$ is smooth in y. In this case

$$\begin{aligned}v_3(t,x) &= \int_0^t\int_{\mathbb{R}^n}<\Gamma(t-r,x-\cdot),\tilde g(r,\cdot)>\,dw_r\\ &= \int_0^t\int_{\mathbb{R}^n}\Gamma(t-r,x-y)\tilde g(r,y)dy\,dw_r,\qquad t\in[0,T],\ x\in\mathbb{R}^n.\end{aligned}$$

Define $\bar{v}_3(t,x) = v_3(Tt, \sqrt{T}x)$, $t \in [0,1]$ and $\bar{\tilde{g}}(r,y) = \tilde{g}(Tr, \sqrt{T}y)$, $r \in [0,1]$. Note

$$\begin{aligned}
\bar{v}_3(t,x) &= \int_0^{Tt} \int_{\mathbb{R}^n} \Gamma(Tt-r, \sqrt{T}x-y)\tilde{g}(r,y)dy\, dw_r \\
&= \sqrt{T}\int_0^{t} \int_{\mathbb{R}^n} (\sqrt{T})^n \Gamma(Tt-Tr, \sqrt{T}x-\sqrt{T}y)\bar{\tilde{g}}(r,y)dy\, d\bar{w}_r \\
&= \sqrt{T}\int_0^{t} \int_{\mathbb{R}^n} \Gamma(t-r, x-y)\bar{\tilde{g}}(s,y)dyd\bar{w}_r.
\end{aligned}$$

By obvious scaling and (6.1) we receive

$$\begin{aligned}
& E\int_{\mathbb{R}^n}\int_0^T\int_0^T \frac{|v_3(t,x)-v_3(s,x)|^p}{|t-s|^{1+\frac{p}{2}k}}dsdt\, dx \\
= \quad & T^{1-\frac{p}{2}k+\frac{n}{2}}\, E\int_{\mathbb{R}^n}\int_0^1\int_0^1 \frac{|\bar{v}_3(t,x)-\bar{v}_3(s,x)|^p}{|t-s|^{1+\frac{p}{2}k}}dsdt\, dx. \\
\lesssim \quad & T^{1-\frac{p}{2}k+\frac{n}{2}}\, \|\bar{\tilde{g}}\|^p_{L^p(\Omega\times(0,1),\mathcal{P},B_p^{k-1}(\mathbb{R}^n))}. \qquad (6.2)
\end{aligned}$$

To dominate (6.2) by $\|\tilde{g}\|^p_{L^p(\Omega\times(0,T),\mathcal{P},B_p^{k-1}(\mathbb{R}^n))}$ we observe the following. Given a smooth function $f=f(y)$ define $f_{\sqrt{T}}(y) = f\left(\sqrt{T}\,y\right)$. Then for any $\phi \in C_0^\infty(\mathbb{R}^n)$, $\|\phi\|_{B_q^{1-k}(\mathbb{R}^n)} = 1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}
\int_{\mathbb{R}^n} f_{\sqrt{T}}(y)\phi(y)dy &= T^{-\frac{n}{2}}\int_{\mathbb{R}^n} f(y)\phi_{\frac{1}{\sqrt{T}}}(y)dy \\
&\leq T^{-\frac{n}{2}}\|f\|_{B_p^{k-1}(\mathbb{R}^n)}\,\|\phi_{\frac{1}{\sqrt{T}}}\|_{B_q^{1-k}(\mathbb{R}^n)} \\
&\leq T^{-\frac{n}{2}}\|f\|_{B_p^{k-1}(\mathbb{R}^n)}\cdot T^{\frac{n}{2}}(1\vee T^{-p(1-k)})\,\|\phi\|_{B_q^{1-k}(\mathbb{R}^n)} \\
&\leq (1\vee T^{-p(1-k)})\,\|f\|_{B_p^{k-1}(\mathbb{R}^n)};
\end{aligned}$$

see Remark 2.1 (2) for the second inequality. Hence, $\|f_{\sqrt{T}}\|^p_{B_p^{k-1}(\mathbb{R}^n)} \leq (1\vee T^{k-1})\,\|f\|^p_{B_p^{k-1}(\mathbb{R}^n)}$. This and another simple scaling imply that (6.2) is indeed bounded by $c\|\tilde{g}\|^p_{L^p(\Omega\times(0,T),\mathcal{P},B_p^{k-1}(\mathbb{R}^n))}$, where c depends only on p,n,k,T.

We need two more lemmas to prove Lemma 3.7 (1) with $T=1$. The proof of the following lemmas are placed at the end of this section.

Lemma 6.1. *Let $\frac{1}{p}<k<1$, $p\geq 2$ and $\tilde{g}\in\mathbb{H}_p^{k-1}(\mathbb{R}_1^n)$. Then for $i=-1,-2,\dots$ we have*

$$E\int_{\mathbb{R}^n}\int\int_{4^i\leq|t-s|\leq 4^{i+1}} \frac{|v_3(t,x)-v_3(s,x)|^p}{|t-s|^{1+\frac{p}{2}k}}dsdt\, dx \lesssim \|\tilde{g}\|^p_{L^p(\Omega\times(0,1),\mathcal{P},H_p^{k-1}(\mathbb{R}^n))}. \qquad (6.3)$$

Let X_0 and X_1 be a couple of Banach spaces continuously embedded in a topological vector space and let Y_0 and Y_1 be another such couple. We denote the real interpolation spaces

$$X_{\theta q} := (X_0,X_1)_{\theta,q},\;\; Y_{\theta q} := (Y_0,Y_1)_{\theta,q}, \quad 0<\theta<1, \quad 1\leq q\leq\infty \qquad (6.4)$$

and the following well known result (see Theorem 1.3 in [18]):

Lemma 6.2. *Let $T=\sum_{-\infty}^{\infty}T_i$, where $T_i : X_\nu \to Y_\nu$ are bounded linear operators with norms $M_{i,\nu}$ such that $M_{i,\nu}\leq c\omega^{i(\theta-\nu)}, \nu=0,1$, for some fixed $\omega\neq 1$ and $0<\theta<1$. Then $T: X_{\theta 1}\to Y_{\theta\infty}$ is a bounded linear operator.*

Let us denote $S\tilde{g} := v_3$.

Proof of Lemma 3.7 (1) 1. As we discussed, it is enough to consider the case $T = 1$. Recall $\frac{1}{p} < k < 1$ and $p \geq 2$. Note that the extension $\tilde{g}$ of g is in $L^p(\Omega \times (0,1), \mathcal{P}, B_p^{k-1}(\mathbb{R}^n))$. Since the random function $S\tilde{g}$ belongs to $\mathbb{B}_p^k(\mathbb{R}_1^n)$ and satisfies (3.9) (Lemma 3.4), to prove (3.11) we only need to show

$$E \int_{\mathbb{R}^n} \int_0^1 \int_0^1 \frac{|S\tilde{g}(t,x) - S\tilde{g}(s,x)|^p}{|t-s|^{1+\frac{p}{2}k}} dsdt\, dx \lesssim \|\tilde{g}\|^p_{L^p(\Omega \times (0,1), \mathcal{P}, B_p^{k-1}(\mathbb{R}^n))} \tag{6.5}$$

; see (2.5) and the time version of Remark 2.1 (2). We follows the outline of [9].

2. Define the space Y whose element $h : \Omega \times (0,1)^2 \times \mathbb{R}^n \to \mathbf{C}$ satisfies

$$\|h\|_Y^p := E \int_{\mathbb{R}^n} \int_0^1 \int_0^1 \frac{|h(t,s,x)|^p}{|t-s|} dsdt\, dx < \infty.$$

Let $\frac{1}{p} < \alpha_1 < k < \alpha_2 < 1$. Denote

$$X_\nu = L^p(\Omega \times (0,1), \mathcal{P}, H_p^{\alpha_\nu - 1}(\mathbb{R}^n)), \quad Y_\nu = Y, \quad \nu = 1, 2$$

and define the operators $T_i : X_\nu \to Y_\nu$ $(i = -1, -2, \ldots)$ by

$$T_i\tilde{g}(\omega, t, s, x) = \begin{cases} \frac{S\tilde{g}(\omega,t,x) - S\tilde{g}(\omega,s,x)}{|t-s|^{\frac{1}{2}k}}, & \text{if } 4^i \leq |t-s| < 4^{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, using Lemma 6.1, we have

$$\|T_i\tilde{g}\|_{Y_\nu} \lesssim 2^{i(\alpha_\nu - k)} \|\tilde{g}\|_{X_\nu}, \quad \nu = 1, 2, \quad i = -1, -2, \ldots.$$

As we take $\theta = \frac{k - \alpha_1}{\alpha_2 - \alpha_1}$ and $\gamma = 2^{\alpha_1 - \alpha_2}$, the norms $M_{i,\nu}$ of the map $T_i : X_\nu \to Y_\nu$ satisfy

$$M_{i\nu} \lesssim 2^{i(\alpha_\nu - k)} = c\gamma^{i(\theta - \nu)}.$$

Note that $Y_{\theta\infty} = Y$. Hence, by Lemma 6.2 we have

$$E \int_{\mathbb{R}^n} \int \int_{|t-s|<1} \frac{|S\tilde{g}(t,x) - S\tilde{g}(s,x)|^p}{|t-s|^{1+\frac{p}{2}k}} dsdt\, dx \lesssim \|\tilde{g}\|^p_{X_{\theta 1}}, \tag{6.6}$$

where

$$X_{\theta 1} \quad := \quad \left(L^p(\Omega \times (0,1), \mathcal{P}, H_p^{\alpha_1 - 1}(\mathbb{R}^n)), L^p(\Omega \times (0,1), \mathcal{P}, H_p^{\alpha_2 - 1}(\mathbb{R}^n))\right)_{\theta,1}.$$

3. Now, choose k_1, k_2 and set $\eta \in (0,1)$ so that

$$\frac{1}{p} < \alpha_1 < k_1 < k < k_2 < \alpha_2 < 1, \qquad k = (1-\eta)k_1 + \eta k_2.$$

Denote $\theta_\mu = \frac{k_\mu - \alpha_1}{\alpha_2 - \alpha_1}$, $\mu = 1, 2$. Then (6.6) holds for the quadruples $(\alpha_1, k_1, \alpha_2, \theta_1)$ and $(\alpha_1, k_2, \alpha_2, \theta_2)$. By Theorem 3.11.5 in [3] and lemma 3.3 we have

$$\begin{aligned}(X_{\theta_1 1}, X_{\theta_2 1})_{\eta,p} &= (L^p(\Omega \times (0,1), \mathcal{P}, H_p^{\alpha_0 - 1}(\mathbb{R}^n)), L^p(\Omega \times (0,1), \mathcal{P}, H_p^{\alpha_1 - 1}(\mathbb{R}^n)))_{\theta,p} \\ &= L^p(\Omega \times (0,1), \mathcal{P}, (H_p^{\alpha_0 - 1}(\mathbb{R}^n), H_p^{\alpha_1 - 1}(\mathbb{R}^n)_{\theta,p}) \\ &= L^p(\Omega \times (0,1), \mathcal{P}, B_p^{k-1}(\mathbb{R}^n)).\end{aligned}$$

On the other hand define the weights on $d\pi := dPdtds\,dx$ by

$$w_\mu = w_\mu(\omega,t,s,x) = \frac{1}{|t-s|^{1+\frac{p}{2}k_\mu}}, \mu = 1,2, \quad w = w_1^{1-\eta}\, w_2^\eta.$$

Then by Theorem 5.4.1 (Stein-Weiss interpolation theorem) in [3] we have

$$\big(L^p(\Omega\times(0,1)^2\times\mathbb{R}^n, w_1 d\pi), L^p(\Omega\times(0,1)^2\times\mathbb{R}^n, w_2 d\pi)\big)_{\eta,p} = L^p(\Omega\times(0,1)^2\times\mathbb{R}^n, w\, d\pi).$$

Hence, we receive (6.5). Lemma 3.7 (1) now follows. □

Now, we prove Lemma 6.1. We need the followings. Recall that $\mathcal{S}(\mathbb{R}^n)$ is dense in any $\mathcal{B}_p^k(\mathbb{R}^n)$.

Lemma 6.3. *Let $l<0$, $1<q<\infty$ and $g\in\mathcal{S}(\mathbb{R}^n)$. Then the followings hold.*

(1) For $t>0$,

$$\left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\Gamma(t,x-y)g(y)dy\right|^q dx\right)^{1/q} \lesssim (1+t^{\frac{l}{2}})\|g\|_{H_p^l(\mathbb{R}^n)}.$$

(2) For t, $h>0$,

$$\left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\Big(\Gamma(t+h,x-y)-\Gamma(t,x-y)\Big)g(y)dy\right|^q dx\right)^{1/q} \lesssim h(t^{-1}+t^{\frac{l}{2}-1})\|g\|_{H_p^l(\mathbb{R}^n)}.$$

Proof. (1) Denote $\mathcal{F}(h)=\hat{h}$, the spatial Fourier transform of h. We observe that

$$\mathcal{F}(\Gamma(t,\cdot)*g)(\xi) = (1+|\xi|^{-l})e^{-t|\xi|^2}\cdot m(\xi)(1+|\xi|^2)^{\frac{l}{2}}\widehat{g}(\xi),$$

where $m(\xi)=\frac{(1+|\xi|^2)^{-l/2}}{1+|\xi|^{-l}}$. We note that m is an L^q-Fourier multiplier, i.e., the operator T_m defined by $\widehat{T_m(f)}(\xi)=m(\xi)\hat{f}(\xi)$ is L^q-bounded. On the other hand we set

$$\widehat{K^t}(\xi) = (1+|\xi|^{-l})e^{-t|\xi|^2}.$$

Since $\|\mathcal{F}^{-1}(\widehat{\phi}(\sqrt{t}\xi))\|_1 = \|\phi\|_1$, we obtain

$$\|K^t\|_1 \le \|\mathcal{F}^{-1}(e^{-t|\xi|^2})\|_1 + t^{\frac{l}{2}}\|\mathcal{F}^{-1}((t|\xi|^2)^{-\frac{l}{2}}e^{-t|\xi|^2})\|_1 \lesssim (1+t^{\frac{l}{2}}).$$

We have

$$\Gamma(t,\cdot)*g = K^t*(T_m(I-\Delta)^{\frac{l}{2}}g).$$

By Young's inequality and the multiplier theorem, we conclude that for $1<q<\infty$

$$\left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\Gamma(t,x-y)g(y)dy\right|^q dx\right)^{1/q} \lesssim (1+t^{\frac{l}{2}})\|(I-\Delta)^{\frac{l}{2}}g\|_q = (1+t^{\frac{l}{2}})\|g\|_{H_p^l(\mathbb{R}^n)}.$$

(2) We set

$$\mathcal{F}((\Gamma(t+h,\cdot)-\Gamma(t,\cdot))*g)(\xi) = (-h|\xi|^2)(1+|\xi|^{-l})e^{-t|\xi|^2}\cdot\frac{1-e^{-h|\xi|^2}}{h|\xi|^2}\cdot m(\xi)(1+|\xi|^2)^{\frac{l}{2}}\widehat{g}(\xi),$$

where $m(\xi)=\frac{(1+|\xi|^2)^{-l/2}}{1+|\xi|^{-l}}$. Note that $\frac{1-e^{-h|\xi|^2}}{h|\xi|^2}$ is the L^q-Fourier multiplier and the norm is independent of h. Set

$$\widehat{K^{t,h}}(\xi) = (-h|\xi|^2)(1+|\xi|^{-l})e^{-t|\xi|^2}.$$

Then we have $\|K^{t,h}\|_1 \lesssim h(t^{-1}+t^{\frac{l}{2}-1})$ and the rest is similar to the case (1). □

Lemma 6.4. *Let $\frac{1}{p} < k < 1$. Fix $i = -1, -2, \dots$ and denote $D_i := \{(s,t) \in (0,1) \times (0,1) \,|\, 4^i \leq t - s < 4^{i+1}\}$. Consider the following operators T_1, T_2, T_3 which map function defined on $(0,1)$ to a function defined on D_i:*

$$\begin{aligned}(T_1 f)(s,t) &:= \int_s^t (t-r)^{k-1} f(r)dr, \\ (T_2 f)(s,t) &:= \int_{(s-4^i)\vee 0}^{s} (s-r)^{k-1} f(r)dr, \qquad\qquad (s,t) \in D_i \\ (T_3 f)(s,t) &:= \int_0^{(s-4^i)\vee 0} (s-r)^{k-3} f(r)dr\end{aligned}$$

; note that $T_2 f$ and T_3, in fact, are independent of t. Then for $1 \leq q < \infty$ we have

$$\|T_m f\|_{L^q(D_i)} \leq c_m\, 4^{i(k+\frac{1}{q})} \|f\|_{L^q(0,1)},\ m = 1,2 \quad ; \quad \|T_3 f\|_{L^q(D_i)} \leq c_3\, 4^{i(k-2+\frac{1}{q})} \|f\|_{L^q(0,1)}, \tag{6.7}$$

where c_1, c_2, c_3 are absolute constants.

Proof. 1. For $q = 1$ Fubini's theorem gives us

$$\begin{aligned}\|T_1 f\|_{L^1(D_i)} &= \int_{4^i}^1 \int_{(t-4^{i+1})\vee 0}^{t-4^i} \int_s^t (t-r)^{k-1} |f(r)| dr ds dt \\ &\leq \int_0^1 |f(r)| \left[\int_r^{r+4^{i+1}} (t-r)^{k-1} \left(\int_{t-4^{i+1}}^r ds \right) dt \right] dr \\ &\leq \frac{4^{k+1}}{k(k+1)} \cdot 4^{i(k+1)} \|f\|_{L^1(0,1)}.\end{aligned}$$

For $q = \infty$ we have

$$\sup_{(s,t)\in D_i} |(T_1 f)(s,t)| \leq \|f\|_{L^\infty(0,1)} \cdot \sup_{(s,t)\in D_i} \int_s^t (t-r)^{k-1} dr \leq \frac{4^k}{k} \cdot 4^{ik} \|f\|_{L^\infty(0,1)}.$$

Then, by the real interpolation theorem $(L_1, L_\infty)_{\theta,q} = L_q$ with the relation $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{\infty} = \theta$, we get

$$\|T_1 f\|_{L^q(D_i)} \leq c\, (4^{i(k+1)})^\theta (4^{ik})^{1-\theta}\, \|f\|_{L^q(0,1)} \leq c\, 4^{i(k+\frac{1}{q})}\, \|f\|_{L^q(0,1)}.$$

hence, (6.7) for T_1 holds .

2. For $q = 1$ we have

$$\begin{aligned}\|T_2 f\|_{L^1(D_i)} &= \int_{4^i}^1 \int_{(t-4^{i+1})\vee 0}^{t-4^i} \int_{(s-4^i)\vee 0}^{s} (s-r)^{k-1} |f(r)| dr ds dt \\ &\leq \int_0^1 |f(r)| \left[\int_r^{r+4^i} (s-r)^{k-1} \left(\int_{s+4^i}^{s+4^{i+1}} dt \right) ds \right] dr \\ &\leq \frac{3}{k} \cdot 4^{i(1+k)} \|f\|_{L^1(0,1)}\end{aligned}$$

and for $q = \infty$

$$\sup_{(s,t)\in D_i} |(T_2 f)(s,t)| \leq \|f\|_{L^\infty(0,1)} \cdot \sup_{(s,t)\in D_i} \int_{(s-4^i)\vee 0}^{s} (s-r)^{k-1} dr \leq \frac{1}{k} \cdot 4^{ki}\, \|f\|_{L^\infty(0,1)}.$$

By the real interpolation theorem, (6.7) for T_2 holds.

3. For T_3 the proof is similar. Observe that

$$\begin{aligned}\|T_3 f\|_{L^1(D_i)} &= \int_{4^i}^1 \int_{(t-4^{i+1})\vee 0}^{t-4^i} \int_{(s-4^i)\vee 0}^{s} (s-r)^{k-3}|f(r)|drdsdt \\ &\le \int_0^1 |f(r)| \left[\int_{r+4^i}^1 (s-r)^{k-3} \left(\int_{s+4^i}^{s+4^{i+1}} dt \right) ds \right] dr \\ &\le \frac{3}{2-k} \cdot 4^{i(k-1)} \|f\|_{L^1(0,1)}.\end{aligned}$$

and

$$\sup_{(s,t)\in D_i} |(T_3 f)(s,t)| \le \|f\|_{L^\infty(0,1)} \cdot \sup_{(s,t)\in D_i} \int_0^{(s-4^i)\vee 0} (s-r)^{k-3} dr \le \frac{1}{2-k} \cdot 4^{i(k-2)} \, \|f\|_{L^\infty(0,1)}.$$

By the real interpolation theorem, (6.7) for T_3 holds. $\square$

Proof of Lemma 6.1 1. Fix $i = -1, -2, \ldots$. Since

$$\begin{aligned}&E \int_{\mathbb{R}^n} \int \int_{4^i < |t-s| < 4^{i+1}} \frac{|v_3(t,x) - v_3(s,x)|^p}{|t-s|^{1+\frac{p}{2}k}} dsdtdx \\ &\quad \le 2E \int_{\mathbb{R}^n} \int_0^1 \int_{t-4^{i+1}}^{t-4^i} \frac{|v_3(t,x) - v_3(s,x)|^p}{(t-s)^{1+\frac{p}{2}k}} dsdtdx, \end{aligned} \tag{6.8}$$

we assume $t > s$. Note that

$$\begin{aligned} v_3(t,x) - v_3(s,x) &= \textstyle\int_s^t \int_{\mathbb{R}^n} \Gamma(t-r, x-y) \tilde{g}(r,y) dy \, dw_r \\ &\quad + \textstyle\int_0^s \int_{\mathbb{R}^n} (\Gamma(t-r, x-y) - \Gamma(s-r, x-y)) \tilde{g}(r,y) dy \, dw_r. \end{aligned} \tag{6.9}$$

The right-hand side of (6.8) is bounded by the sum of the following quantities (up to a constant multiple):

$$I_1 = E \int_{\mathbb{R}^n} \int_0^1 \int_{(t-4^{i+1})\vee 0}^{(t-4^i)\vee 0} \frac{|\int_s^t \int_{\mathbb{R}^n} \Gamma(t-r, x-y) \tilde{g}(r,y) dy dw_r|^p}{(t-s)^{1+\frac{p}{2}k}} \, dsdtdx,$$

$$I_2 = E \int_{\mathbb{R}^n} \int_0^1 \int_{(t-4^{i+1})\vee 0}^{(t-4^i)\vee 0} \frac{|\int_0^s \int_{\mathbb{R}^n} (\Gamma(t-r, x-y) - \Gamma(s-r, x-y)) \tilde{g}(r,y) dy dw_r|^p}{(t-s)^{1+\frac{p}{2}k}} \, dsdtdx.$$

2. Recall that we assume $\frac{1}{p} < k < 1$ and $p \ge 2$.

<u>Estimation of I_1</u> By Burkholder-Davis-Gundy inequality(BDG) (see Section 2.7 in [12]) I_1 is dominated by, up to a constant multiple,

$$E \int_{\mathbb{R}^n} \int_{4^i}^1 \int_{(t-4^{i+1})\vee 0}^{t-4^i} \frac{(\int_s^t |\int_{\mathbb{R}^n} \Gamma(t-r, x-y) \tilde{g}(r,y) dy|^2 dr)^{\frac{p}{2}}}{(t-s)^{1+\frac{p}{2}k}} \, dsdtdx. \tag{6.10}$$

Next, by Minkowski's inequality for integrals and Lemma 6.3 (1), the expression (6.10) is bounded

by, up to a constant multiple,

$$
\begin{aligned}
& E\int_{4^i}^1\int_{(t-4^{i+1})\vee 0}^{t-4^i}\frac{(\int_s^t(\int_{\mathbb{R}^n}|\int_{\mathbb{R}^n}\Gamma(t-r,x-y)\tilde{g}(r,y)dy|^p dx)^{\frac{2}{p}}dr)^{\frac{p}{2}}}{(t-s)^{1+\frac{p}{2}k}}dsdt \\
\lesssim\ & E\int_{4^i}^1\int_{(t-4^{i+1})\vee 0}^{t-4^i}\frac{(\int_s^t(t-r)^{k-1}\|\tilde{g}(r,\cdot)\|^2_{H_p^{k-1}(\mathbb{R}^n)}dr)^{\frac{p}{2}}}{(t-s)^{1+\frac{p}{2}k}}dsdt \\
\lesssim\ & 4^{-i(1+\frac{p}{2}k)}E\int_{4^i}^1\int_{(t-4^{i+1})\vee 0}^{t-4^i}\left(\int_s^t(t-r)^{k-1}\|\tilde{g}(r,\cdot)\|^2_{H_p^{k-1}(\mathbb{R}^n)}dr\right)^{\frac{p}{2}}dsdt.
\end{aligned}
$$

Applying Lemma 6.4 with the operator T_1 and $\frac{p}{2}$ in place of q, we receive

$$I_1\lesssim c\,\|\tilde{g}\|^p_{L^p(\Omega\times(0,1),\mathcal{P},H_p^{k-1}(\mathbb{R}^n))}.$$

Estimation of I_2 BDG inequality I_2 is dominated by, up to a constant multiple,

$$
\begin{aligned}
& E\int_{\mathbb{R}^n}\int_{4^i}^1\int_{(t-4^{i+1})\vee 0}^{t-4^i}\frac{(\int_0^s|\int_{\mathbb{R}^n}(\Gamma(t-r,x-y)-\Gamma(s-r,x-y))\tilde{g}(r,y)dy|^2dr)^{\frac{p}{2}}}{(t-s)^{1+\frac{p}{2}k}}dsdtdx \\
\lesssim\ & E\int_{\mathbb{R}^n}\int_{4^i}^1\int_{(t-4^{i+1})\vee 0}^{t-4^i}\frac{(\int_{(s-4^i)\vee 0}^s|\int_{\mathbb{R}^n}(\Gamma(t-r,x-y)-\Gamma(s-r,x-y))\tilde{g}(r,y)dy|^2dr)^{\frac{p}{2}}}{(t-s)^{1+\frac{p}{2}k}}dsdtdx \\
& +E\int_{\mathbb{R}^n}\int_{4^i}^1\int_{(t-4^{i+1})\vee 0}^{t-4^i}\frac{(\int_0^{(s-4^i)\vee 0}|\int_{\mathbb{R}^n}(\Gamma(t-r,x-y)-\Gamma(s-r,x-y))\tilde{g}(r,y)dy|^2dr)^{\frac{p}{2}}}{(t-s)^{1+\frac{p}{2}k}}dsdtdx \\
=\ & I_{21}+I_{22}.
\end{aligned}
$$

By Minkowski's inequality for integrals and Lemma 6.3 (1) the term I_{21} is bounded by, up to a constant multiple,

$$
\begin{aligned}
& E\int_{4^i}^1\int_{(t-4^{i+1})\vee 0}^{t-4^i}\frac{(\int_{(s-4^i)\vee 0}^s((t-r)^{k-1}+(s-r)^{k-1})\|\tilde{g}(r,\cdot)\|^2_{H_p^{k-1}(\mathbb{R}^n)}dr)^{\frac{p}{2}}}{(t-s)^{1+\frac{p}{2}k}}dsdt \\
\lesssim\ & 4^{-i(1+\frac{p}{2}k)}E\int_{4^i}^1\int_{(t-4^{i+1})\vee 0}^{t-4^i}\left(\int_{(s-4^i)\vee 0}^s(s-r)^{k-1}\|\tilde{g}(r,\cdot)\|^2_{H_p^{k-1}(\mathbb{R}^n)}dr\right)^{\frac{p}{2}}dsdt
\end{aligned}
$$

; we used $k<1$. Lemma 6.4 with the operator T_1 gives us

$$I_{21}\lesssim c\,\|\tilde{g}\|^p_{L^p(\Omega\times(0,1),\mathcal{P},H_p^{k-1}(\mathbb{R}^n))}.$$

By Minkowski's inequality for integrals again and Lemma 6.3 (2) the term I_{22} is dominated by, up to a constant multiple,

$$
\begin{aligned}
& E\int_{4^i}^1\int_{(t-4^{i+1})\vee 0}^{t-4^i}\frac{(\int_0^{(s-4^i)\vee 0}(\int_{\mathbb{R}^n}|\int_{\mathbb{R}^n}(\Gamma(t-r,x-y)-\Gamma(s-r,x-y))\tilde{g}(r,y)dy|^p dx)^{\frac{2}{p}}dr)^{\frac{p}{2}}}{(t-s)^{1+\frac{p}{2}k}}dsdt \\
\lesssim\ & 4^{-i(1+\frac{p}{2}k)}E\int_{4^i}^1\int_{(t-4^{i+1})\vee 0}^{t-4^i}\left(\int_0^{(s-4^i)\vee 0}(t-s)^2(s-r)^{k-3}\|\tilde{g}(r,\cdot)\|^2_{H_p^{k-1}(\mathbb{R}^n)}dr\right)^{\frac{p}{2}}dsdt \\
\lesssim\ & 4^{-i(1+\frac{p}{2}k)}\cdot 4^{ip}\cdot E\int_{4^i}^1\int_{(t-4^{i+1})\vee 0}^{t-4^i}\left(\int_0^{(s-4^i)\vee 0}(s-r)^{k-3}\|\tilde{g}(r,\cdot)\|^2_{H_p^{k-1}(\mathbb{R}^n)}dr\right)^{\frac{p}{2}}dsdt.
\end{aligned}
$$

Then Lemma 6.4 with the operator T_3 gives us

$$I_{22} \lesssim c\,\|\tilde{g}\|^p_{L^p(\Omega\times(0,1),\mathcal{P},H^{k-1}_p(\mathbb{R}^n))}.$$

3. By the estimations of I_1, I_2 our claim (6.3) follows. $\square$

7 Proof of Lemma 3.7 (2)

Again, we just assume $T = 1$. We start with the following lemmas.

Lemma 7.1. *For $0 < t,\ r < \infty$*

$$\int_{\mathbb{R}^n} |\Gamma(t+r,y) - \Gamma(r,y)|dy \lesssim \begin{cases} \frac{t}{r}, & t < r, \\ 1, & t \geq r. \end{cases}$$

; this is almost obvious and the proof is omitted.

Lemma 7.2. *Let $0 \leq \theta < 1,\ 1 < p < \infty$. Then for $g \in H^\theta_{p,o}(D)$,*

$$\int_D \delta(y)^{-p\theta}|g(y)|^p dy \leq c\|g\|^p_{H^\theta_{p,o}(D)},$$

where $\delta(y) = dist(y, \partial D)$. The constant c depends only on p, n.

Proof. We may assume $0 < \theta < 1$. We use complex interpolation of L^p-spaces of measures. Let $d\mu_0(y) = dy$ and $d\mu_1(y) = \delta^{-p}(y)dy$. The complex interpolation space between $L^p(d\mu_0)$ and $L^p(d\mu_1)$ with index θ is

$$(L^p(d\mu_0), L^p(d\mu_1))_{[\theta]} = L^p(d\mu_\theta), \quad d\mu_\theta(y) := \delta^{-p\theta}dy$$

(see Theorem 5.5.3 in [3]). Note that using Hardy's inequality, we obtain that for $g \in H^1_{p,o}(D)$

$$\left(\int_D \delta(y)^{-p}|g(y)|^p dy\right)^{\frac{1}{p}} \leq c\left(\int_D |\nabla g(y)|^p dy\right)^{\frac{1}{p}} = c\,\|g\|_{H^1_{p,o}(D)}.$$

Since $(H^1_{p,o}(D), L^p(D))_{[\theta]} = H^\theta_{p,o}(D)$ (see Proposition 2.1 in [7]), we get

$$\left(\int_D \delta^{-p\theta}(y)|g(y)|^p dy\right)^{\frac{1}{p}} \leq c\,\|g\|_{(H^1_{p,o}(D),L^p(D))_{[\theta]}} = c\,\|g\|_{H^\theta_{p,o}(D)}.$$

$\square$

Proof of Lemma 3.7 (2) 1. Recall $1 \leq k < 1 + \frac{1}{p}$ and $p \geq 2$. For $g \in \mathbb{H}^{k-1}_p(D_T) = L^p(\Omega\times(0,1),\mathcal{P},H^{k-1}_{p,o}(D))$, we denote $\tilde{g} \in L^p(\Omega\times(0,1),\mathcal{P},H^{k-1}_p(\mathbb{R}^n))$ by $\tilde{g}(\omega,t,x) = g(\omega,t,x)$ for $x \in D$ and $\tilde{g}(\omega,t,x) = 0$ for $x \in \mathbb{R}^n\setminus\bar{D}$. Then by lemma 3.4, we have $v_3 \in L^p(\Omega\times(0,1),\mathcal{P},H^k_p(\mathbb{R}^n))$, where

$$v_3(t,x) = \int_0^t \int_{\mathbb{R}^n} \Gamma(t-s,x-y)\tilde{g}(s,y)dy\,dw_s.$$

By the usual trace theorem (see [9]), we get $v_3|_{\partial D_T} \in L^p(\Omega \times (0,1), \mathcal{P}, B_p^{k-\frac{1}{p}}(\partial D))$. Hence, it is sufficient to show that

$$E \int_{\partial D} \int\int_{0<s<t<1} \frac{|v_3(x,t) - v_3(x,s)|^p}{(t-s)^{1+\frac{p}{2}(k-\frac{1}{p})}} dsdt\ d\sigma(x) \lesssim \|g\|_{L^p((0,1),\mathcal{P},H_p^{k-1}(D))}. \tag{7.1}$$

Then, using real interpolation (see lemma 3.3), we complete the proof of lemma 3.7 (2).

2. The left-hand side of (7.1) is bounded by the sum of the following quantities (up to a constant multiple):

$$J_1 = E \int_{\partial D} \int_0^1 \int_0^t \frac{|\int_s^t \int_D \Gamma(t-r, x-y)\tilde{g}(r,y) dy dw_r|^p}{(t-s)^{1+\frac{p}{2}(k-\frac{1}{p})}} dsdtd\sigma(x),$$

$$J_2 = E \int_{\partial D} \int_0^1 \int_0^t \frac{|\int_0^s \int_D (\Gamma(t-r, x-y) - \Gamma(s-r, x-y))\tilde{g}(r,y) dy dw_r|^p}{(t-s)^{1+\frac{p}{2}(k-\frac{1}{p})}} dsdtd\sigma(x).$$

<u>Estimation of J_1</u> By BDG's inequality, J_1 is dominated by, up to a constant multiple,

$$E \int_{\partial D} \int_0^1 \int_0^t \frac{(\int_s^t |\int_D \Gamma(t-r, x-y)\tilde{g}(r,y) dy|^2 dr)^{\frac{p}{2}}}{(t-s)^{1+\frac{p}{2}(k-\frac{1}{p})}} dsdtd\sigma(x). \tag{7.2}$$

Note

$$\begin{aligned}
&\int_{\partial D} \Big(\int_s^t |\int_D \Gamma(t-r, x-y)\tilde{g}(r,y) dy|^2 dr \Big)^{\frac{p}{2}} d\sigma(x) \\
\lesssim\ & \Big(\int_s^t \Big(\int_{\partial D} |\int_D \Gamma(t-r, x-y)\tilde{g}(r,y) dy|^p d\sigma(x) \Big)^{\frac{2}{p}} dr \Big)^{\frac{p}{2}} \\
\lesssim\ & \Big(\int_s^t \Big(\int_{\partial D} (\int_{\mathbb{R}^n} \Gamma(t-r, x-y) dy)^{\frac{p}{p'}} \cdot \int_D \Gamma(t-r, x-y) |\tilde{g}(r,y)|^p dy\ d\sigma(x) \Big)^{\frac{2}{p}} dr \Big)^{\frac{p}{2}} \\
\lesssim\ & \Big(\int_s^t \Big(\int_D |\tilde{g}(r,y)|^p\ \int_{\partial D} \Gamma(t-r, x-y) d\sigma(x)\ dy \Big)^{\frac{2}{p}} dr \Big)^{\frac{p}{2}},
\end{aligned} \tag{7.3}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Note that for $y \in D$ there is a $x_y \in \partial D$ such that $\delta(y) = |y - x_y|$, where $\delta(y) = dist(y, \partial D)$. Since D is a bounded Lipschitz domain, there is $r_0 > 0$ independent of x_y such that $|y - x| \approx \delta(y) + |x - x_y|$ for all $|x - x_y| < r_0$. We have

$$\begin{aligned}
&\int_{\partial D} \Gamma(t-r, x-y) d\sigma(x) \\
\lesssim\ & \int_{|x-x_y|<r_0} \Big[(t-r)^{-\frac{n}{2}} \cdot e^{-c\frac{\delta(y)^2 + |x-x_y|^2}{t-r}} \Big] d\sigma(x) + \int_{|x-x_y|\geq r_0} \Big[(t-r)^{-\frac{n}{2}} \cdot e^{-c\frac{\delta(y)^2 + |x-x_y|^2}{t-r}} \Big] d\sigma(x) \\
\lesssim\ & \int_{|x'|<r_0,\, x' \in \mathbb{R}^{n-1}} \Big[(t-r)^{-\frac{n}{2}} \cdot e^{-c\frac{\delta(y)^2 + |x'|^2}{t-r}} \Big] dx' + (t-r)^{-\frac{n}{2}} \cdot e^{-c\frac{\delta(y)^2 + r_0^2}{t-r}} \\
\lesssim\ & (t-r)^{-\frac{1}{2}} \cdot e^{-c\frac{\delta(y)^2}{t-r}} \Big[\int_{\mathbb{R}^{n-1}} e^{-c|y'|^2} dy' + (t-r)^{\frac{n-1}{2}} \cdot e^{-c\frac{r_0^2}{t-r}} \Big] \\
\lesssim\ & (t-r)^{-\frac{1}{2}} \cdot e^{-c\frac{\delta(y)^2}{t-r}}.
\end{aligned} \tag{7.4}$$

By (7.4) and the Hölder inequality, the last term in (7.3) is bounded by, up to a constant multiple,

$$\Big(\int_s^t \Big(\int_D |\tilde g(r,y)|^p (t-r)^{-\frac12} e^{-c\frac{\delta(y)^2}{t-r}}dy\Big)^{\frac2p} dr\Big)^{\frac p2} \lesssim (t-s)^{\frac{p-2}{2}} \int_s^t \int_D |\tilde g(r,y)|^p (t-r)^{-\frac12} e^{-c\frac{\delta(y)^2}{t-r}} dydr. \tag{7.5}$$

Hence, via Fubini's Theorem, (7.2) is dominated by, up to a constant multiple,

$$\begin{aligned}
&E\int_0^1\int_D |\tilde g(r,y)|^p \left[\int_r^1\int_0^r (t-s)^{-\frac p2(k-1)-\frac32}\frac{1}{(t-r)^{\frac12}}e^{-c\frac{\delta(y)^2}{t-r}}dsdt\right]dydr\\
&\lesssim E\int_0^1\int_D |\tilde g(r,y)|^p\left[\int_r^1 e^{-c\frac{\delta(y)^2}{t-r}}\cdot (t-r)^{-\frac p2(k-1)-1}dt\right]dydr\\
&= E\int_0^1\int_D |\tilde g(r,y)|^p\left[\int_0^{1-r} e^{-c\frac{\delta(y)^2}{t}}\cdot t^{-\frac p2(k-1)-1}dt\right]dydr\\
&= E\int_0^1\int_D \delta^{-p(k-1)}(y)|\tilde g(r,y)|^p\left[\int_{\frac{\delta^2(y)}{1-r}}^\infty e^{-ct}\cdot t^{\frac p2(k-1)-1}dt\right]dydr\\
&\lesssim E\int_0^1\int_D \delta^{-p(k-1)}(y)|\tilde g(r,y)|^p dydr\\
&\lesssim E\int_0^1 \|g(\cdot,r)\|^p_{H^{k-1}_{p,o}(D)}dr
\end{aligned}$$

; for the last inequality we used the assumption $g\in\mathbb{H}^{k-1}_{p,o}(D)$ and Lemma 7.2 with $\theta=k-1$.

Estimation of J_2 By BDG's inequality, J_2 is dominated by, up to a constant multiple,

$$E\int_{\partial D}\int_0^1\int_0^t \frac{(\int_0^s|\int_D(\Gamma(t-r,x-y)-\Gamma(s-r,x-y))\tilde g(r,y)dy|^2dr)^{\frac p2}}{(t-s)^{1+\frac p2(k-\frac1p)}}dsdtd\sigma(x). \tag{7.6}$$

Define $A := A(t,s,r,x,y) = \Gamma(t-r,x-y)-\Gamma(s-r,x-y)$. If $p>2$, using the Hölder inequality twice, we get

$$\begin{aligned}
\Big(\int_0^s\Big|\int_D A\cdot\tilde g(r,y)dy\Big|^2dr\Big)^{\frac p2} &\le \Big(\int_0^s\Big[\int_D|A|dy\Big]^{\frac{2(p-1)}{p}}\Big[\int_D|A||\tilde g(r,y)|^pdy\Big]^{\frac2p}dr\Big)^{\frac p2}\\
&\le \Big(\int_0^s\Big[\int_D|A|dy\Big]^{\frac{2(p-1)}{p-2}}dr\Big)^{\frac{p-2}{2}}\int_0^s\int_D|A||\tilde g(r,y)|^pdydr.
\end{aligned} \tag{7.7}$$

Next, by changing variable from r to $s-r$ and Lemma 7.1,

$$\begin{aligned}
\int_0^s\Big(\int_D|A|dy\Big)^{\frac{2(p-1)}{p-2}}dr &= \int_0^s\Big(\int_D|\Gamma(t-s+r,x-y)-\Gamma(r,x-y)|dy\Big)^{\frac{2(p-1)}{p-2}}dr\\
&\lesssim \begin{cases}\int_0^s dr, & s<t-s\\ \int_0^{t-s}dr+\int_{t-s}^s(\frac{t-s}{r})^{\frac{2(p-1)}{p-2}}dr, & s\ge t-s\end{cases}\\
&= \begin{cases} s, & s<t-s\\ t-s+(t-s)^{\frac{2(p-1)}{p-2}}((t-s)^{-\frac{p}{p-2}}-s^{-\frac{p}{p-2}}) & s\ge t-s\end{cases}\\
&\lesssim (t-s)
\end{aligned}$$

and

$$\Big(\int_0^s\Big|\int_D A\cdot\tilde g(r,y)dy\Big|^2dr\Big)^{\frac p2}\lesssim (t-s)^{\frac{p-2}{2}}\int_0^s\int_D|A||\tilde g(r,y)|^pdydr. \tag{7.8}$$

If $p=2$, (7.7) with $p=2$ and Lemma 7.1 immediately yields (7.8). Hence, (7.6) is dominated by, up to a constant multiple,

$$
\begin{aligned}
& E\int_{\partial D}\int_0^1\int_0^t\left[\int_0^s\int_D |A(t,s,r,x,y)||\tilde{g}(r,y)|^p dydr\right](t-s)^{-\frac{p}{2}(k-1)-\frac{3}{2}}dsdt\, d\sigma(x) \\
\lesssim\ & E\int_0^1\int_D|\tilde{g}(r,y)|^p\left[\int_0^{1-r}\int_0^t(t-s)^{-\frac{p}{2}(k-1)-\frac{3}{2}}\int_{\partial D}|\Gamma(t,x-y)-\Gamma(s,x-y)|d\sigma(x)\, dsdt\right]dydr.
\end{aligned}
\tag{7.9}
$$

We estimate the boundary (∂D) integral part: Since $s<t$, we have

$$
\begin{aligned}
&\int_{\partial D}|\Gamma(t,x-y)-\Gamma(s,x-y)|d\sigma(x) \\
&\le \left(\frac{1}{s^{\frac{1}{2}n}}-\frac{1}{t^{\frac{1}{2}n}}\right)\int_{\partial D}e^{-\frac{|x-y|^2}{4s}}d\sigma(x)+t^{-\frac{1}{2}n}\int_{\partial D}e^{-\frac{|x-y|^2}{4t}}-e^{-\frac{|x-y|^2}{4s}}d\sigma(x) \\
&= K_1+K_2.
\end{aligned}
$$

Applying (7.4) again,

$$
\begin{aligned}
K_1=\frac{t^{\frac{1}{2}n}-s^{\frac{1}{2}n}}{t^{\frac{1}{2}n}s^{\frac{1}{2}n}}\int_{\partial D}e^{-\frac{|x-y|^2}{4s}}d\sigma(x) &\le \frac{t^{\frac{1}{2}n}-s^{\frac{1}{2}n}}{t^{\frac{1}{2}n}}s^{-\frac{1}{2}}e^{-c\frac{\delta(y)^2}{s}} \\
&\le \begin{cases} s^{-\frac{1}{2}}e^{-c\frac{\delta(y)^2}{s}}, & 0<s<\frac{1}{2}t, \\ t^{-\frac{3}{2}}(t-s)e^{-c\frac{\delta(y)^2}{t}}, & \frac{1}{2}t\le s<t. \end{cases}
\end{aligned}
$$

For K_2 we consider two cases. If $0<s<\frac{1}{2}t$, using (7.4), we get

$$
K_2\le t^{-\frac{1}{2}n}\int_{\partial D}e^{-\frac{|x-y|^2}{4t}}d\sigma(x)\le t^{-\frac{1}{2}}e^{-c\frac{\delta(y)^2}{t}}.
$$

For $\frac{1}{2}t<s<t$, using the Mean Value Theorem, there is a η satisfying $s<\eta<t$ such that

$$
K_2=t^{-\frac{1}{2}n}\int_{\partial D}(t-s)\frac{|x-y|^2}{4\eta^2}e^{-\frac{|x-y|^2}{\eta}}d\sigma(x)
$$

and this leads to

$$
\begin{aligned}
K_2 &\lesssim t^{-\frac{1}{2}n}\int_{\partial D}(t-s)\frac{|x-y|^2}{t^2}e^{-\frac{|x-y|^2}{4t}}d\sigma(x) \\
&\lesssim t^{-\frac{1}{2}n-2}(t-s)\int_{\partial D}(|x-x_y|^2+\delta(y)^2)e^{-c\frac{|x-x_y|^2+\delta(y)^2}{t}}d\sigma(x) \\
&\lesssim t^{-\frac{1}{2}n-2}(t-s)e^{-c\frac{\delta(y)^2}{t}}(t^{\frac{n+1}{2}}+\delta^2(y)t^{\frac{n-1}{2}}) \\
&= (t-s)e^{-c\frac{\delta(y)^2}{t}}(t^{-\frac{3}{2}}+\delta(y)^2t^{-\frac{5}{2}}).
\end{aligned}
$$

By these estimations, the bracket in (7.9) is bounded by, up to a constant multiple,

$$
\begin{aligned}
&\int_0^{1-r} t^{-\frac{p}{2}(k-1)-\frac{3}{2}} \left[\int_0^{\frac{1}{2}t} (t^{-\frac{1}{2}}e^{-c\frac{\delta(y)^2}{t}} + s^{-\frac{1}{2}}e^{-c\frac{\delta(y)^2}{s}})ds\right] dt \\
&+\int_0^{1-r} e^{-c\frac{\delta(y)^2}{t}}(t^{-\frac{3}{2}}+\delta(y)^2 t^{-\frac{5}{2}}) \left[\int_{\frac{1}{2}t}^{t} (t-s)^{-\frac{p}{2}(k-1)-\frac{1}{2}}ds\right] dt \\
&\lesssim \int_0^{1-r} \left[t^{-\frac{p}{2}(k-1)-1}e^{-c\frac{\delta(y)^2}{t}} + t^{-\frac{p}{2}(k-1)-\frac{3}{2}}\cdot\delta(y)\cdot\int_{\frac{2\delta(y)^2}{t}}^{\infty} s^{-\frac{3}{2}}e^{-cs}ds + t^{-\frac{p}{2}(k-1)-2}\cdot e^{-c\frac{\delta(y)^2}{t}}\cdot\delta(y)^2\right] dt \\
&\quad =: L_1+L_2+L_3
\end{aligned}
$$

; for the inequality we used the assumption $k<1+\frac{1}{p}$. It is easy to see that the terms L_1 and L_3 are dominated by $\delta(y)^{-p(k-1)}$. This is also true for L_2; if $2\delta(y)^2 \geq (1-r)$, then $2\delta(y)^2\geq t$ and

$$
L_2 \lesssim \delta(y)\int_0^{1-r} t^{-\frac{p}{2}(k-1)-\frac{3}{2}}\int_{\frac{2\delta(y)^2}{t}}^{\infty} s^{-\frac{3}{2}}e^{-cs}dsdt \lesssim \delta(y)\int_0^{1-r} t^{-\frac{p}{2}(k-1)-\frac{3}{2}}e^{-c\frac{\delta(y)^2}{t}}dt \lesssim \delta(y)^{-p(k-1)}.
$$

If $2\delta(y)^2\leq 1-r$,

$$
\begin{aligned}
L_2 &\lesssim \delta(y)\int_0^{1-r} t^{-\frac{p}{2}(k-1)-\frac{3}{2}}\int_{\frac{2\delta(y)^2}{t}}^{\infty} s^{-\frac{3}{2}}e^{-cs}dsdt \\
&\lesssim \delta(y)\int_0^{2\delta(y)^2} t^{-\frac{p}{2}(k-1)-\frac{3}{2}}e^{-c\frac{\delta(y)^2}{t}}dt + \int_{2\delta(y)^2}^{1-r} t^{-\frac{p}{2}(k-1)-1}dt \\
&\lesssim \delta(y)^{-p(k-1)}.
\end{aligned}
$$

After all, (7.9) (hence J_2) is bounded by, up to a constant multiple,

$$
E\int_0^1\int_D \delta(y)^{-p(k-1)}|\tilde{g}(r,y)|dydr \lesssim E\int_0^1 \|g(\cdot,r)\|^p_{H^{k-1}_p(D)}dr
$$

; we used the assumption $g\in\mathbb{H}^{k-1}_{p,o}(D)$ and Lemma 7.2.

3. The step 2 implies (7.1). The lemma is proved. □

References

[1] R. Brown, *The initial-Neumann problem for the heat equation in Lipschitz cylinders*, Trans. Amer. Math. Soc, **320**, no. 1, 1-52 (1990).

[2] R. Brown, *The method of layer potentials for the heat equation in Lipschitz cylinders*, Amer. J. Math., **111**, no. 2, 339-379 (1989).

[3] J. Bergh and J. Lofström, *Interpolation spaces, An introduction*, Springer-Verlag, Berlin(1976).

[4] T. Chang, *Extension and Restriction theorems in time variable bounded domains*, Commun. Contemp. Math., **12**, no. 2, 265-294(2010).

[5] E. Fabes and N. Riviere, *Dirichlet and Neumann problems for the heat equation in C^1-cylinders*, Harmonic analysis in Euclidean spaces, part 2, 179-196, Proc. Sympos. Pure Math., XXXV, Part, Amer. Math. Soc., Providence, RI, 1979.

[6] T. Jakab and M. Mitrea, *Parabolic initial boundary value problems in nonsmooth cylinders with data in anisotropic Besov spaces*, Math. Res. Lett, **13**, no. 5-6, 825-831(2006).

[7] D. Jerison and C. Kenig, *The inhomogeneous Dirichlet Problem in Lipschitz domains*, J. of Funct. Anal, **130**, 161-219(1995).

[8] B.F. Jones Jr., *Lipschitz spaces and the heat equation*, J. Math. Mech., **18**, 379-409 (1968).

[9] A. Jonsson and H. Wallin, *A whitney extension theorem in L_p and besov spaces*, Ann.Inst. Fourier Grenoble, **28**, 139-192 (1978).

[10] K. Kim, *On stochastic partial differential equations with variable coefficients in C^1 domains*, Stochastic processes and their applications, **112**, no. 2, 261-283 (2004).

[11] K. Kim and N.V. Krylov, *On SPDEs with variable coefficients in one space dimension*, Potential Anal, **21**, no. 3, 203-239 (2004).

[12] N.V. Krylov, *An analytic approach to SPDEs*, Stochastic partial differential equations: six perspectives, Math. Surveys Monogr., **64**, 185-242, Amer.Math.Soc., Providence, RI, 1999.

[13] N.V. Krylov, *A generalization of the Littlewood-Paley inequality and some other results related to stochastic partial differential equations*, Ulam Quart., **2**, no. 4, 16-26 (1994).

[14] N.V. Krylov and S.V. Lototsky, *A Sobolev space theory of SPDEs with constant coefficients on a half line*, SIAM J. Math. Anal., **30**, no. 2, 298-325 (1999).

[15] N.V. Krylov and S.V. Lototsky, *A Sobolev space theory of SPDEs with constant coefficients in a half space*, SIAM J. on Math. Anal., **31**, no. 1, 19-33 (1999).

[16] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural′ceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs, **23**, American Mathematical Society, Providence, R.I. 1967.

[17] S.V. Lototsky, *Dirichlet problem for stochastic parabolic equations in smooth domains*, Stochastics and Stochastics Reports, **68**, no. 1-2, 145-175 (1999).

[18] J. Peetre, *On the trace of potentials*, Ann. Scuola Norm. Sup. Pisa, **2**,1,33-43(1975).

[19] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton. N.J, 1970.

[20] H. Triebel, *Theory of function spaces*, Monographs in Mathematics **78**, Birkhauser Verlag, Basel, 1983.